SOCIEDADE BRASILEIRA DE MATEMÁTICA

ENSAIOS MATEMÁTICOS
2014, Volume **26**, 1–77

# Global divergence theorems in nonlinear PDEs and geometry

**Stefano Pigola**
**Alberto G. Setti**

**Abstract.** These lecture notes contain, in a slightly expanded form, the material presented at the Summer School in Differential Geometry held in January 2012 in the Universidade Federal do Ceará-UFC, Fortaleza.

The course aims at giving an overview of some $L^p$-extensions of the classical divergence theorem to non-compact Riemannian manifolds without boundary. The red wire connecting all these extensions is represented by the notion of parabolicity with respect to the $p$-Laplace operator. It is a non-linear differential operator which is naturally related to the $p$-energy of maps and, therefore, to $L^p$-integrability properties of vector fields. To show the usefulness of these tools, a certain number of applications both to (systems of) PDEs and to the global geometry of the underlying manifold are presented.

**2010 Mathematics Subject Classification:** 31C12, 31C15, 31C45, 53C20, 53C21, 58E20.

# Acknowledgments

The authors are grateful to the Department of Mathematics of the Universidade Federal do Ceará-UFC for the invitation to participate in the Summer School and for the very warm hospitality. A special thank goes to G. Pacelli Bessa, Luquésio Jorge and Jorge H. de Lira. The first author is also grateful to Giona Veronelli for suggestions and helpful discussions related to the topics of the paper.

# Contents

**1 The divergence theorem in the compact setting: introductory examples 7**
1.1 Introduction . . . 7
1.2 Basic notation . . . 8
1.2.1 Manifold and curvature tensors . . . 8
1.2.2 Metric objects and their measures . . . 9
1.2.3 Differential operators . . . 9
1.2.4 Function spaces . . . 10
1.3 The divergence theorem in the compact setting and regularity remarks . . . 11
1.4 Divergence theorem and solutions of (nonlinear) PDEs . . . 13
1.5 From real-valued functions to manifold-valued maps: divergence theorem and systems of PDEs . . . 17
1.5.1 Harmonic maps . . . 17
1.5.2 $p$-harmonic maps . . . 21
1.5.3 Concluding remarks . . . 24

**2 Extending the divergence theorem to non-compact manifolds 26**
2.1 Some potential theory for the $p$-Laplacian . . . 26
2.1.1 Parabolicity & maximum principles . . . 30
2.1.2 Parabolicity & capacity . . . 33
2.1.3 Global divergence theorems for $L^q(q > 1)$-vector fields: the Kelvin-Nevanlinna-Royden criterion . . . 36
2.2 A notion of $\infty$-parabolicity . . . 38
2.2.1 Liouville & spectral consequences of $\infty$-parabolicity 41
2.2.2 Global divergence theorem for $L^1$-vector fields: Gaffney theorem . . . 45
2.3 Global divergence theorems: relaxing the $L^q$-assumptions . 46
2.3.1 Karp version of the $L^1$-divergence theorem . . . 46
2.3.2 An improved version of the $L^q(q > 1)$-divergence theorem . . . 47

**3 Some applications of the global divergence theorems 50**
3.1 The case of $L^q(q > 1)$-vector fields . . . . . . . . . . . . . . . . . 50
3.1.1 Global comparison principles . . . . . . . . . . . . . . 50
3.1.2 Concluding remarks on comparison & Liouville theory 55
3.2 The case of $L^1$-vector fields . . . . . . . . . . . . . . . . . . . . . . 55
3.2.1 Liouville theorem for harmonic functions of sublinear growth . . . . . . . . . . . . . . . . . . . . . . . . . 55
3.2.2 Lower volume estimates & rigidity for the sharp Sobolev constant . . . . . . . . . . . . . . . . . . . . . . 57

**A On the optimality of cut-off functions 64**

**B The strong comparison principle for graphs 66**

**C The maximum of subsolutions is a subsolution 69**

**Bibliography 73**

# Chapter 1

# The divergence theorem in the compact setting: introductory examples

## 1.1 Introduction

On a closed Riemannian manifold, the usual divergence theorem, or integration by parts, is an invaluable tool to obtain integral estimates of solutions of PDEs, corresponding vanishing results, comparison theories etc. which, in turn, can also be used to deduce information on the geometry and the topology of the underlying space. A prototypical example is represented by the Bochner technique and its geometric consequences. In noncompact settings the usual divergence theorem requires objects to be compactly supported. Therefore, we can agree that this is a local result which does not take into account the global (say at infinity) geometry of the underlying space. A global version should involve vector fields (and differential forms) with prescribed asymptotic behavior at infinity, such as an $L^p$ condition. The validity of a corresponding Stokes theorem then reflects the fact that the manifold has a negligible boundary at infinity from the viewpoint of $L^p$ vector fields and this is intimately related to potential theoretic properties (parabolicity) of the space. Thus, in the parabolic realm, the integration by parts technique extends globally and gives new information and results on the behavior of solutions of PDEs as well as geometric implications on the global geometry of the space. These lectures aim at exploring this order of ideas. We shall discuss the abstract framework and present some concrete examples where the general theory yields interesting results.

## 1.2 Basic notation

### 1.2.1 Manifold and curvature tensors

1. $(M, \langle , \rangle)$ is a connected Riemannian manifold, $\dim M = m$, $\partial M = \varnothing$.

2. $D$ is Levi-Civita connection of $M$.

3. The Riemann $(1,3)$-tensor is
$$\mathrm{Riem}\,(X,Y)\,Z = [D_X, D_Y]\,Z - D_{[X,Y]}Z.$$

4. The Riemann $(0,4)$-tensor is
$$\mathrm{Riem}\,(X,Y,Z,W) = \langle \mathrm{Riem}\,(X,Y)\,Z, W\rangle\,.$$

5. The sectional curvature is defined by
$$\mathrm{Sec}\,(X \wedge Y) = \frac{\langle \mathrm{Riem}\,(X,Y)\,Y, X\rangle}{|X \wedge Y|^2}.$$

6. The radial sectional curvature with respect to an origin $o \in M$ is
$$\mathrm{Sec}_{\mathrm{rad}}\,(X) = \mathrm{Sec}\,(X \wedge \nabla r)\,,$$
where $r\,(x) = d\,(x,o)$.

7. The Ricci $(1,1)$-tensor is
$$\begin{aligned}\mathrm{Ric}\,(X) &= \mathrm{tr}\,\{(Z,W) \to \mathrm{Riem}\,(X,Z)\,W\} \\ &= \sum_{i=1}^{m} \mathrm{Riem}\,(X,E_i)\,E_i.\end{aligned}$$

8. The Ricci $(0,2)$-tensor is
$$\mathrm{Ric}\,(X,Y) = \langle \mathrm{Ric}\,(X)\,, Y\rangle\,.$$

9. The scalar curvature is defined by
$$\mathrm{Scal} = \mathrm{tr}\,(X \to \mathrm{Ric}\,(X)) = \sum_{i=1}^{m} \mathrm{Ric}\,(E_i, E_i)\,.$$

### 1.2.2 Metric objects and their measures

1. The intrinsic distance of $M$ is denoted by $d(x,y)$ or $d_M(x,y)$. It is realized as the infimum of the length of (rectifiable) paths connecting $x$ and $y$.

2. The geodesic ball centered at $o \in M$ and of radius $r$ is denoted by
$$B_r(o) = \{x \in M : d(x,o) < r\}.$$
Its boundary is the geodesic sphere
$$\partial B_r(0) = \{x \in M : d(x,o) = r\}.$$

3. The Riemannian metric gives rise to a canonical measure $d\mathrm{vol}_m$ which, in local coordinates, is given by
$$d\mathrm{vol}_m = \sqrt{G}dx^1 \cdots dx^m,$$
where $G = \det(g_{ij})$. The corresponding $(m-1)$-Hausdorff measure is $d\mathrm{vol}_{m-1}$. Set
$$\mathrm{vol}B_r = \int_{B_r} d\mathrm{vol}_m$$
and
$$\mathrm{area}\partial B_r = \int_{\partial B_r} d\mathrm{vol}_{m-1}.$$
By the co-area formula
$$\mathrm{vol}B_r = \int_0^r \mathrm{area}\partial B_r dr.$$
When there is no danger of confusion, in writing integrals, the relevant measures will be often understood.

4. The volume growth of $M$ is the growth rate of the function $\mathrm{vol}B_r$ whereas the area growth is the grow rate of the function $\mathrm{area}\partial B_r$.

5. The cut-locus of a point $o \in M$ is denoted by $\mathrm{cut}(o)$. It follows that $r_o(x) = d(x,o)$ is a smooth function on $M \backslash (\mathrm{cut}(o) \cup \{o\})$.

### 1.2.3 Differential operators

Let $u : M \to \mathbb{R}$ be a given smooth (say $C^2$) function

1. The gradient of $u$ is the vector field $\nabla u$ defined by
$$\langle \nabla u, X \rangle = du(X).$$
In local coordinates
$$\nabla u = g^{ij}\partial_j u \; \partial_i,$$
where $g_{ij} = \langle \partial_i, \partial_j \rangle$.

2. The Hessian of $u$ is the bilinear form
$$\operatorname{Hess}(u)(X,Y) = \langle D_X \nabla u, Y\rangle .$$
In local coordinates
$$\operatorname{Hess}(u)_{ij} = \partial^2_{ij} u - \Gamma^l_{ij}\partial_l u,$$
with $\Gamma^l_{ij}$ the Christoffel symbols of $D$.
3. The divergence of the vector field $X$ is defined by
$$\operatorname{div} X = \operatorname{tr}(Y \to D_Y X).$$
In local coordinates, if $X = X^i \partial_i$ then
$$\begin{aligned}\operatorname{div} X &= \frac{1}{\sqrt{G}}\partial_i\left(\sqrt{G}X^i\right)\\ &= \partial_i X^i + X^t \Gamma^i_{it}\end{aligned}$$
with $G = \det(g_{ij})$.
4. The Laplacian (Laplace-Beltrami operator) of $u$ is
$$\Delta u = \operatorname{trHess} = \operatorname{div}(\nabla u).$$
In particular, if $M = \mathbb{R}$, $\Delta = d^2/dx^2$. In local coordinates
$$\begin{aligned}\Delta u &= \frac{1}{\sqrt{G}}\partial_i\left(\sqrt{G}g^{ik}\partial_k u\right)\\ &= g^{ik}\partial^2_{ik}u - g^{ik}\Gamma^t_{ik}\partial_t u.\end{aligned}$$

Further definitions and notation (e.g. the Laplacian of maps) will be given when needed.

### 1.2.4 Function spaces

The symbol $Lip(M)$ stands for the space of Lipschitz functions, i.e., according to Rademacher theorem, the space of continuous functions which are differentiable a.e. and have bounded gradient. The space $W^{1,p}(M)$ is defined as the space of $L^p$-functions whose distributional gradients are in $L^p$. It is a Banach space when equipped with the norm $\|\varphi\|_{W^{1,p}} = \|\varphi\|_{L^p} + \|\nabla\varphi\|_{L^p}$. Adding the index "*loc*" to function spaces means that the defining properties of the space hold on every compact set $K \subset M$. Let $C^k_c(M)$ denote the space of compactly supported, $k$-differentiable functions on $M$. The closure of $C^\infty_c(M)$ in $W^{1,p}(M)$ is denoted by $W^{1,p}_0(M)$.

## 1.3 The divergence theorem in the compact setting and regularity remarks

The usual divergence theorem states that

**Theorem 1.1.** *Let $X$ be a smooth, compactly supported vector field on $M$. Then*

$$\int_M \operatorname{div} X = 0.$$

*In particular this holds for every smooth vector field on a closed manifold $M$.*

Sometimes it is necessary to relax the regularity conditions on $X$ and $\operatorname{div} X$. This is possible provided we interpret the divergence of an $L^1_{loc}$-vector field in the sense of distributions.

**Definition 1.2.** *Let $X \in L^1_{loc}(M)$. The distributional divergence of $X$ is defined by*

$$(\varphi, \operatorname{div} X) = -\int_M \langle X, \nabla\varphi\rangle, \ \forall \varphi \in C^\infty_c(M). \tag{1.1}$$

**Remark 1.3.** *If $X \in L^q(M)$, then by density arguments the previous definition extends to every $\varphi \in W_0^{1,\frac{q}{q-1}}(M)$.*

When $\operatorname{div} X \in L^1_{loc}(M)$

$$(\varphi, \operatorname{div} X) = \int_M \varphi \operatorname{div} X$$

and, therefore,

$$\int_M \varphi \operatorname{div} X = -\int_M \langle X, \nabla\varphi\rangle.$$

If, in addition, $X$ is compactly supported we can choose $\varphi = 1$ on the support of $X$ and recover the usual equality

$$\int_M \operatorname{div} X = 0.$$

Moreover, assuming again that $\operatorname{div} X \in L^1_{loc}(M)$, the following standard equality holds for every $\rho \in Lip_{loc}(M)$:

$$\operatorname{div}(\rho X) = \langle X, \nabla\rho\rangle + \rho \operatorname{div} X. \tag{1.2}$$

Indeed, for every $\varphi \in C^\infty_c(M)$

$$\begin{aligned}(\varphi, \operatorname{div}(\rho X)) &= -\int_M \langle X, \rho\nabla\varphi\rangle \\ &= -\int_M \langle X, \nabla(\rho\varphi)\rangle + \int_M \langle X, \varphi\nabla\rho\rangle \\ &= \int_M (\rho \operatorname{div} X + \langle X, \nabla\rho\rangle)\varphi \\ &= (\varphi, \rho \operatorname{div} X + \langle X, \nabla\rho\rangle).\end{aligned}$$

In particular, $\operatorname{div}(\rho X) \in L^1_{loc}$ and formula (1.2) holds true. This is nothing but a different manifestation of the divergence theorem (or integration by parts).

**Claim 1.4** (A). *According to the above observations, if* $X \in L^1_{loc}(M)$ *and* $\operatorname{div} X \in L^1_{loc}(M)$, *then computations like that in formula (1.2) can be formally carried over as in the smooth case. Moreover, if* $X$ *is compactly supported, then the formal divergence theorem applies. We will tacitly do that in all that follows.*

Obviously, integration by parts is also related to the notion of distributional solution of differential inequalities in situations of low regularity.

**Definition 1.5.** *Let* $X \in L^1_{loc}(M)$ *be a given vector field and* $f \in L^1_{loc}(M)$ *a given function. We say that*

$$\operatorname{div} X \geq f, \text{ on } M$$

*in the sense of distributions, if the following inequality*

$$(\varphi, \operatorname{div} X) \geq (f, \varphi),$$

*holds for every* $0 \leq \varphi \in C^\infty_c(M)$, *namely,*

$$-\int_M \langle X, \nabla\varphi \rangle \geq \int f\varphi. \tag{1.3}$$

*The definitions of distributional solution of* $\operatorname{div} X \leq f$ *is completely similar.*

Note that, according to Definition 1.2 above, a function $f \in L^1_{loc}$ is the distributional derivative of the vector field $X$, provided (1.1) holds for every $\varphi \in C^\infty_c(M)$. However, it turns out that it suffices that (1.1) hold for every nonnegative test function. Indeed, if this is the case it follows that $(\varphi, \operatorname{div} X - f) = 0$ for every $0 \leq \varphi \in C^\infty_c(M)$, and $\operatorname{div} X - f$ is a nonnegative distribution, hence a positive locally finite measure, and, since it is also nonpositive, such measure is necessarily the zero measure. In particular, if $\operatorname{div} X \geq f$ and $\operatorname{div} X \leq f$ then $\operatorname{div} X = f$.

**Remark 1.6.** *As above, if* $X \in L^q(M)$ *and* $f \in L^q(M)$ *then the definition extends to every* $0 \leq \varphi \in W_0^{1,\frac{q}{q-1}}(M)$.

Note that, this time, we do not assume $\operatorname{div} X \in L^1_{loc}(M)$. However this is not a serious restriction with respect to formal manipulations. Indeed, what we shall often need is that from an inequality of the type $\operatorname{div} X \geq f \geq 0$ for some compactly supported vector field $X \in L^q(M)$, one is able to deduce e.g. that $\int_M f \leq 0$. The correct procedure is to use the distributional definition (1.3) and choose $\varphi = 1$ on the support of

$X$. However, from the viewpoint of the final goal, nothing is lost if we apply formally the divergence theorem. One may suspect that, this time, a problem arises when it is needed to compute $\operatorname{div}(\rho X)$ for some function $0 \le \rho \in W^{1,\frac{q}{q-1}}_{loc}(M)$. Namely, from $\operatorname{div} X \ge f$ we would like to obtain that $-\int_M \langle X, \nabla\rho\rangle \ge \int_M \rho f$. Again, the correct procedure is to apply the distributional definition of $\operatorname{div} X \ge f$ with the test function $\varphi = \rho\eta$ for some $0 \le \eta \in C^\infty_c(M)$. This gives $-\int_M \langle X, \nabla\rho\rangle \eta - \int_M \langle X, \nabla\eta\rangle \rho \ge \int_M \rho\eta f$ and, assuming that $X$ has compact support, we choose $\eta = 1$ on the support of $X$ (note that, necessarily, $f \le 0$ outside the support of $X$). However, as before, the formal application of the divergence theorem to $Z = \rho X$ produces the same effect.

**Claim 1.7** (B)**.** *On the ground of the above observations, in the presence of an inequality of the type* $\operatorname{div} X \ge f$*, we shall use formal manipulations relying on the divergence theorem and (1.2) even when* $\operatorname{div} X$ *is only a distribution and not an* $L^1_{loc}$*-function.*

Actually, this claim extends even if $X$ is not compactly supported but, in this case, we have to assume some extra global integrability property on $X$ and some property on $M$ that guarantee the existence of special tests functions. This will be the content of the global extensions of the divergence theorem to the non-compact realm. See Remark 2.21 in Chapter 2.

## 1.4 Divergence theorem and solutions of (nonlinear) PDEs

To begin with we recall the following elementary

**Theorem 1.8.** *Let $M$ be a closed manifold. Then every function $w \in W^{1,2}_{loc}(M) \cap C^0(M)$ satisfying $\Delta w \ge 0$ (or $\Delta w \le 0$) must be constant.*

**Definition 1.9.** *A function $w \in W^{1,2}_{loc}(M) \cap C^0(M)$ satisfying $\Delta w \ge 0$ in the distributional sense is called subharmonic. Reversing the inequality, i.e. $\Delta w \le 0$, we get a superharmonic function. If the inequality is replaced by the equality $\Delta w = 0$ then $w$ is smooth by elliptic regularity and it is called a harmonic function.*

**Proof.** By a translation we can always assume that $w \ge 0$. Let $X = w\nabla w$ and, using the subharmonicity of $w$, we compute, $\operatorname{div} X = w\Delta w + |\nabla w|^2 \ge |\nabla w|^2$. By the divergence theorem, we get

$$\int_M |\nabla w|^2 = 0,$$

hence $w$ is constant. ■

Note that the above result is a comparison between the subharmonic function $w$ and the constant (superharmonic) function $c = w(x_0)$. Because of the linearity of the Laplace-Beltrami operator, we obviously have

**Corollary 1.10.** *Let $M$ be a closed manifold and let $u, v : M \to \mathbb{R}$ be solutions of $\Delta u \geq \Delta v$. Then $u(x) - v(x) = \mathrm{const}$.*

**Proof.** Take $w = u - v$ and apply Theorem 1.8. ∎

It is clear that the Stokes-oriented proof of above constancy result works essentially without changes even if we consider non-linear operators in divergence form. However, comparing solutions of differential inequalities involving these operators deserves more attention. According to the terminology introduced by Rigoli-Setti, [41], we put the following

**Definition 1.11.** *The $\varphi$-Laplacian of a function $u$ is the nonlinear, divergence form operator defined by*

$$L_{\varphi}(u) = \mathrm{div}\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right),$$

*where $\varphi \in C^0([0, +\infty)) \cap C^1((0, +\infty))$ satisfies the following structural conditions:*

$$(i)\ \varphi(0) = 0,\ (ii)\ \varphi(t) > 0\ \forall t > 0,\ (iii)\ \varphi(t) \leq A t^{p-1},$$

*for some constants $A, p > 1$.*

**Example 1.12.** *The $\varphi$-Laplacian encompasses very different type of nonlinearities. Indeed, important choices are:*

1. *$\varphi(t) = t^{p-1}$, $1 < p < +\infty$, corresponding to the usual p-Laplace operator*

$$\Delta_p u = \mathrm{div}\left(|\nabla u|^{p-2} \nabla u\right).$$

*The structure and the potential theory of this operator will be of basic importance in extending the Stokes theorem to non-compact manifolds. Note that, if $p = 2$, the p-Laplacian is nothing but the Laplace-Beltrami operator.*

2. *$\varphi(t) = \frac{t}{\sqrt{1+t^2}}$, corresponding to the mean curvature operator*

$$H_u = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right).$$

*The name comes from the fact that the mean curvature of the graph $\Gamma_u(x) = (x, u(x)) : M \to M \times \mathbb{R}$ with respect to the downward pointing* *unit normal* $\mathcal{N} = (\nabla u, -1)/\sqrt{1+|\nabla u|^2}$ *is given by* $-H_u/m$. *It is also important to remark that, from the viewpoint of the structural conditions satisfied by* $\varphi$, *the mean curvature operator (non-linear) and the Laplace-Beltrami operator (linear) have the same behavior!*

We have the following very general comparison result

**Theorem 1.13.** *Keeping the above notation, assume that* $\varphi$ *satisfies the further structural condition*

$$(iv)\ \varphi(t) \text{ is strictly increasing.}$$

*If* $u, v \in W^{1,p}_{loc}(M) \cap C^0(M)$ *satisfy*

$$L_\varphi(u) \geq L_\varphi(v), \text{ on } M,$$

*then* $u - v \equiv \text{const}$.

The further structural assumption required on $\varphi$ can be considered as a strict ellipticity condition and will be used to apply the following simple but crucial

**Lemma 1.14.** *Under the above assumptions on* $\varphi(t)$, *the following holds. Let* $(V, \langle , \rangle)$ *be an* $n$*-dimensional, real vector space endowed with the scalar product* $\langle , \rangle$. *Then, for every* $\xi, \eta \in V$,

$$h(\xi,\eta) := \left\langle |\xi|^{-1}\varphi(|\xi|)\,\xi - |\eta|^{-1}\varphi(|\eta|)\,\eta, \xi - \eta \right\rangle \geq 0,$$

*with equality holding if and only if* $\xi = \eta$.

**Proof (of Lemma 1.14).** Easy manipulations show that

$$\begin{aligned} h(\xi,\eta) = &\{\varphi(|\xi|) - \varphi(|\eta|)\}\{|\xi| - |\eta|\} \\ &+ \left\{|\xi|^{-1}\varphi(|\xi|) + |\eta|^{-1}\varphi(|\eta|)\right\}\{|\xi|\,|\eta| - \langle \xi,\eta\rangle\}. \end{aligned}$$

The first summand is non-negative by the monotonicity of $\varphi$ whereas the second one is non-negative by the Cauchy-Schwartz inequality. Therefore $h(\xi,\eta) \geq 0$. In case the equality holds, then, from the first summand we get $|\xi| = |\eta|$ and the fact that $\xi = \eta$ follows from the equality case in the Cauchy-Schwartz inequality. ■

**Proof (of Theorem 1.13).** Fix any $x_0 \in M$, let $A = u(x_0) - v(x_0)$ and define $\Omega_A$ to be the connected component of the open set

$$\{x \in M : A - 1 < u(x) - v(x) < A + 1\}$$

which contains $x_0$. By standard topological arguments, $\Omega_A \neq \emptyset$ is a (connected) open set. Let $\alpha : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be the piecewise linear function defined by

$$\alpha(t) = \begin{cases} 0 & t \leq A-1, \\ (t-A+1)/2 & A-1 \leq t \leq A+1, \\ 1 & t \geq A+1. \end{cases}$$

Since $\alpha \in Lip(\mathbb{R})$ then $\alpha(u-v) \in W^{1,p}_{loc}(M)$ and $\nabla\alpha(u-v) = \alpha'(u-v)(\nabla u - \nabla v)$. Consider the vector field

$$X = \alpha(u-v)\left\{|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u - |\nabla v|^{-1}\varphi(|\nabla v|)\nabla v\right\}.$$

Computing its divergence we obtain

$$\begin{aligned} \operatorname{div} X &= \alpha'(u-v)\left\langle |\nabla u|^{-1}\varphi(|\nabla u|)\nabla u - |\nabla v|^{-1}\varphi(|\nabla v|)\nabla v, \nabla u - \nabla v\right\rangle \\ &\quad + \alpha(u-v)\left(L_\varphi(u) - L_\varphi(v)\right). \end{aligned}$$

By assumption, the second term is $\geq 0$. On the other hand, by Lemma 1.14,

$$h(\nabla u, \nabla v)(x) = \left\langle |\nabla u|^{-1}\varphi(|\nabla u|)\nabla u - |\nabla v|^{-1}\varphi(|\nabla v|)\nabla v, \nabla u - \nabla v\right\rangle \geq 0,$$

equality holding at some point $x \in M$ if and only if $\nabla u(x) - \nabla v(x) = 0$. In conclusion, using also that $\alpha'(u-v) \geq 0$ and applying the divergence theorem we obtain

$$0 = \int_M \operatorname{div} X \geq \int_M \alpha'(u-v)\, h(\nabla u, \nabla v) \geq 0,$$

proving that

$$\alpha'(u-v)\, h(\nabla u, \nabla v) = 0.$$

Since $\alpha'(u-v) \neq 0$ on $\Omega_A$, we deduce

$$u - v \equiv A, \text{ on } \Omega_A.$$

It follows that the open set $\Omega_A$ is also closed. Since $M$ is connected we must conclude that $\Omega_A = M$ and $u - v = A$ on $M$. ∎

When specified to the $p$-Laplace and to the mean-curvature operators, the abstract comparison principle established in Theorem 1.13 has the following interesting consequences.

**Corollary 1.15.** *If $M$ is closed and*

$$\Delta_p u \geq \Delta_p v, \text{ on } M$$

*then $u - v \equiv \text{const}$. In particular, if $\Delta_p u \geq 0$ then $u$ is constant.*

**Corollary 1.16.** *If the mean curvatures of the graphs* $\Gamma_u$ *and* $\Gamma_v$ *on the closed manifold* $M$ *satisfy*

$$H_u \geq H_v$$

*then* $\Gamma_u$ *and* $\Gamma_v$ *are parallel to each others. In particular, if* $H_u \geq 0$ *then* $\Gamma_u$ *is a slice of* $M \times \mathbb{R}$.

**Remark 1.17.** *It is worth to point out that the same conclusion, for a closed domain* $M$, *can be also achieved using the well known touching principle for graphs which, in turn, is based on a simple linearization argument; see Appendix B.*

# 1.5 From real-valued functions to manifold-valued maps: divergence theorem and systems of PDEs

## 1.5.1 Harmonic maps

In the mid '60s, Eells and Sampson generalized the notion of a harmonic function to the case of manifold-valued maps. This opened an entire new world and gave a completely new way to study the topology of manifolds. The concept of harmonic map requires the introduction of a suitable Laplace operator. From now on, although it is not strictly necessary, we shall assume that manifold-valued maps are smooth. This will simplify the exposition.

**Definition 1.18.** *Let* $u : M \to N$ *be a smooth map. Its differential* $du$ *can be considered as a* $u^{-1}TN$*-valued* 1*-form, i.e., a smooth section of the bundle* $TM^* \otimes u^{-1}TN$. *Let* $\bar{D}$ *denotes the connection induced on this bundle by the Levi-Civita connection* $D$ *of* $M$. *Then* $\bar{D}du \in TM^* \otimes TM^* \otimes u^{-1}TN$ *is the generalized Hessian of* $u$. *Its trace is called the Laplacian, or tension field, of the manifold-valued map* $u$, *and we write*

$$\Delta u = \mathrm{tr}_M \bar{D} du \in u^{-1}TN.$$

*In local coordinates, using the index convention* $A, B, C = 1, ..., n = \dim N$ *and* $i, j, k, ... = 1, ..., m = \dim M$, *we have*

$$(\Delta u)^A = g^{ij} \partial^2_{ij} u^A - g^{ij} \ {}^M\Gamma^k_{ij} \partial_k u^A + \ {}^N\Gamma^A_{BC} g^{ij} \partial_i u^B \partial_j u^C$$

*which can be written in the compact form*

$$(\Delta u)^A = \Delta u^A + \ {}^N\Gamma^A (\partial u, \partial u).$$

*Say that* $u : M \to N$ *is a harmonic map, if* $\Delta u = 0$. *In local coordinates,* $u$ *is harmonic if and only if it is a solution of the differential system*

$$\Delta u^A + \ {}^N\Gamma^A (\partial u, \partial u) = 0, \ \forall A = 1, ..., n.$$

**Example 1.19.** *If* $N = \mathbb{R}^n$ *and* $u = (u^1, ..., u^n)$, *then* $u : M \to \mathbb{R}^n$ *is harmonic iff* $\Delta u^A = 0$, *for every* $A = 1, ..., n$.

**Example 1.20.** *If* $M = \mathbb{R}$, *a harmonic map* $u : \mathbb{R} \to N$ *is a geodesic of* $N$.

**Example 1.21.** *If* $u : M \to N$ *is an isometric immersion, then* $\bar{D}du$ *is the second fundamental form of* $u$ *and* $\Delta u = m\mathbf{H}$, *where* $\mathbf{H}$ *is the mean curvature vector field. In particular, a totally geodesic map is harmonic. Furthermore, the immersion* $u$ *is minimal if and only if* $u$ *is harmonic.*

The simple constancy result recalled in Theorem 1.8 can be readily generalized to harmonic maps into Cartan-Hadamard manifolds. Recall that

**Definition 1.22.** *A Cartan-Hadamard manifold is a complete, simply connected Riemannian manifold of non-positive sectional curvature.*

**Theorem 1.23.** *Let* $u : M \to N$ *be a harmonic map. Suppose that* $M$ *is closed and that* $N$ *is Cartan-Hadamard. Then* $u$ *is constant.*

We need to record the following important facts. The first is the composition law of tension fields.

**Lemma 1.24.** *Let* $u : M \to N$ *and* $\alpha : N \to \mathbb{R}$. *Then*

$$\operatorname{Hess}(\alpha \circ u) = d\alpha|_u \left(\bar{D}du\right) + \operatorname{Hess}(\alpha)|_u (du, du)$$

*and*

$$\Delta(\alpha \circ u) = d\alpha|_u (\Delta u) + \operatorname{tr}_M \operatorname{Hess}(\alpha)|_u (du, du).$$

*In particular, if* $u$ *is harmonic and* $\alpha$ *is a convex function, then* $\alpha \circ u$ *is subharmonic.*

The second result is the comparison principle for the Hessian of the distance function by R. Greene and H.H. Wu, [14]. The name comes from the fact that, in the original formulation, the Hessian of the distance function from a point is compared with the Hessian of the distance function from the pole in a corresponding model manifold; see Definition 1.38 below.

**Lemma 1.25.** *Let* $(M, g)$ *be a complete manifold of dimension* $m$ *and let* $r(x) = d(x, o)$ *be the distance function from a fixed origin* $o \in M$. *Assume that the radial sectional curvature satisfies* $\operatorname{Sec}_{\mathrm{rad}} \leq c$ *with* $c \in \mathbb{R}$. *Set*

$$\operatorname{sn}_c(t) = \begin{cases} \frac{1}{\sqrt{c}} \sin\left(\sqrt{c}t\right) & \text{if } c > 0, \\ t & \text{if } c = 0, \\ \frac{1}{\sqrt{-c}} \sinh\left(\sqrt{-c}t\right) & \text{if } c < 0. \end{cases}$$

*Then, the inequality*

$$\operatorname{Hess}(r) \geq \frac{\operatorname{sn}_c'(r(x))}{\operatorname{sn}_c(r(x))} \{g - dr \otimes dr\}, \tag{1.4}$$

*holds on* $M \backslash \mathrm{cut}(o)$ *in the sense of quadratic forms. Moreover, the equality in (1.4) holds on some ball* $B_{R_0}(o) \subset M \backslash \mathrm{cut}(o)$ *if and only if* $B_{R_0}(o)$ *is isometric to the ball* $\mathbf{B}_{R_0}(0)$ *in the corresponding space-form of* $\mathbf{M}^m(c)$ *of constant curvature* $c$.

*In case* $\mathrm{Sec}_{\mathrm{rad}} \geq c$*, then an estimates of the type (1.4) holds with the reverse inequalities.*

**Remark 1.26.** *Note that, in particular, if we set*

$$\mathrm{in}_c(t) = \int_0^t \mathrm{sn}_c(s)\, ds,$$

*then the curvature inequality* $\mathrm{Sec}_{\mathrm{rad}} \leq c$ *implies*

$$\mathrm{Hess}(\mathrm{in}_c(r(x))) = \mathrm{sn}_c'(r(x))\, dr \otimes dr + \mathrm{sn}_c(r(x))\, \mathrm{Hess}(r) \geq \mathrm{sn}_c'(r(x))\, g. \tag{1.5}$$

*For instance, if* $(M, g)$ *is Cartan-Hadamard, then* $c = 0$, $\mathrm{sn}_c(t) = t$, $\mathrm{sn}_c'(t) = 1$, $\mathrm{in}_c(t) = t^2/2$ *and we get*

$$\mathrm{Hess}\left(r^2\right) \geq 2g, \text{ on } M.$$

Combining these results we get

**Corollary 1.27.** *Let* $u : M \to N$ *be a harmonic map into the Cartan-Hadamard manifold* $N$ *and let* $r_{y_0} = d_N(\cdot, y_0) : N \to \mathbb{R}$. *Then*

$$\Delta\left(r_{y_0}^2 \circ u\right) \geq 2m\,|du|^2 \geq 0 \text{ on } M,$$

*i.e.* $r_{y_0}^2 \circ u : M \to \mathbb{R}$ *is a subharmonic function.*

**Proof (of Theorem 1.8).** Let $y_0 = u(x_0)$ and consider $w = r_{y_0}^2 \circ u$. Since $w$ is subharmonic and $M$ is closed, by Theorem 1.8 , $w$ is constant. Since $w(x_0) = 0$ we conclude that $w \equiv 0$ i.e. $u \equiv y_0$. ■

A standard geometric consequence is the following

**Corollary 1.28.** *There are no closed minimal submanifolds immersed in a Cartan-Hadamard space* $N$.

Actually, as the proof shows, we only need that the image of $u$ is confined into a region supporting a convex function. Recall that

**Definition 1.29.** *A geodesic ball* $B_R(y_0) \subset N$ *is a regular ball if* $B_R(y_0) \cap \mathrm{cut}(y_0) = \varnothing$ *and, setting* $K = \sup_{B_R(y_0)} \mathrm{Sec}$, *it holds* $\sqrt{K}R < \pi/2$.

Clearly, $B_R(y_0)$ is contractible and, by Hessian comparison, the distance function $r_{y_0}^2(y) = d_N^2(y, y_0)$ is smooth and strictly convex on the regular ball $B_R(y_0)$. It follows that:

**Corollary 1.30.** *There are no closed minimal submanifolds immersed into a regular ball of a complete manifold $N$.*

We now introduce some topology on the target. First, some observations are in order.

**Remark 1.31.** *If $N$ is Cartan-Hadamard then $N$ is contractible and every $u : M \to N$ is homotopic to a constant.*

**Remark 1.32.** *If $N$ is not contractible, the conclusion of Theorem 1.23 in general fails to hold. Indeed, consider the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and let* id $: \mathbb{T}^n \to \mathbb{T}^n$ *be the identity map. Then* id *is a non-constant, totally geodesic (hence harmonic) map. Clearly,* id *is not homotopic to a constant.*

These observations suggest the following version of Theorem 1.23.

**Theorem 1.33.** *Let $u : M \to N$ be harmonic with $M$ closed and $N$ complete satisfying $Sec_N \leq 0$. If $u$ is homotopic to a constant, then $u$ is constant.*

**Proof.** Let $x_0 \in M$ be fixed, and let $y_0 = u(x_0)$. We show that $u(x) = y_0$.

Let $P_M : M' \to M$ and $P_N : N' \to N$ be the Riemannian universal coverings of $M$ and $N$, respectively. In particular, $M = M'/\pi_1(M)$ and $N = N'/\pi_1(N)$ where the fundamental groups acts freely and properly as groups of isometric covering transformations. Choose $P_M(x_0') = x_0$ and $P_N(y_0') = u(x_0)$. Now, we lift $u$ to a map $u' : M' \to N'$ such that $u'(x_0') = y_0'$.

(a) Since $P_M$ and $P_N$ are local isometries, $u'$ is harmonic.

(b) By definition, $u'$ is $u_\#(\pi_1)$-equivariant in the sense that

$$u'(\gamma \cdot x') = u_\#(\gamma) \cdot u'(x'),$$

where $u_\# : \pi_1(M) \to \pi_1(N)$ denotes the induced homomorphism.

(c) Since $u$ is homotopically trivial, it follows that $u_\# \equiv 1$ so that $u'$ is constant on the orbits $\pi_1(M) \cdot x'$

Thus, we can define a function $w : M = M'/\pi_1(M) \to \mathbb{R}$ by setting

$$w(x) = d^2_{N'}(u'(x'), y_0'),$$

where $x^{'}$ is any point in the fibre $P_M^{-1}(x)$. Since $N$ has non-positive curvature, $N'$ is Cartan-Hadamard. It follows that $w$ is subharmonic, i.e., $\Delta w \geq 0$. Application of Theorem 1.8 yields $w \equiv 0$ proving that $u' \equiv y_0'$. It follows that $u \equiv y_0$, as claimed. ■

### 1.5.2 *p*-harmonic maps

As in the case of functions, there is a natural "non-linear" version of the Laplacian of a map which is called the $p$-Laplacian. As we shall see in the next lectures, this operator is suitable to take into consideration higher energies of maps and will become very important to develop $L^p$-integration theories in the non-compact setting. As in the previous section, we still assume that manifold-valued maps are smooth. This assumption, especially for $p \neq 2$, is far from being natural. The appropriate regularity is $C^{1,\alpha}$.

**Definition 1.34.** *Let $u : (M,g) \to (N,h)$ be a smooth map and let $1 < p < +\infty$. The p-Laplacian, or p-tension field, of $u$ is the vector field along $u$ defined by*

$$\Delta_p u = \mathrm{tr}_M \bar{D}\left(|du|^{p-2}\, du\right),$$

*where $|du|$ is the Hilbert-Schmidt norm of $du \in \mathrm{Hom}\left(TM, u^{-1}N\right)$. In local coordinates*

$$|du|^2 = g^{ij} h_{AB} \partial_i u^A \partial_j u^B$$

*and*

$$(\Delta_p u)^A = \mathrm{div}\left(|du|^{p-2}\, \nabla u^A\right) + |du|^{p-2}\ {}^N\Gamma^A(\partial u, \partial u).$$

*The map $u$ is said to be p-harmonic if*

$$\Delta_p u = 0,$$

*i.e., in local coordinates, $u$ satisfies the system of equations*

$$\mathrm{div}\left(|du|^{p-2}\, \nabla u^A\right) + |du|^{p-2}\ {}^N\Gamma^A(\partial u, \partial u) = 0,$$

*for $A = 1, ..., n$. In case $u$ is only assumed to be $C^1$, this must be interpreted in the weak form*

$$\int |du|^{p-2}\left\{g^{ij} h_{AB} \partial_i u^A \partial_j \eta^A - {}^N\Gamma^A(\partial u, \partial u)\, h_{AB}\eta^B\right\} = 0.$$

**Remark 1.35.** *It is worth to point out that, even for a map $u : M \to \mathbb{R}^n$, it is not true that $u$ is p-harmonic if and only $\Delta_p u^A = 0$ for every $A = 1, ..., n$. Indeed, the definition of each component $(\Delta_p u)^A$ involves the whole Jacobian of $u$ in the form $|\mathrm{Jac}(u)|^{p-2}$. The p-harmonic system for $u$ reads*

$$\mathrm{div}\left(|\mathrm{Jac}(u)|^{p-2}\, \nabla u^A\right) = 0.$$

We aim at obtaining a $p$-version of Theorems 1.23 and 1.33. We shall prove

**Theorem 1.36.** *Let $u : M \to N$ be a $p$-harmonic map homotopic to a constant, $p \geq 2$. Suppose that $M$ is closed and that $N$ is complete with* $\mathrm{Sec}_N \leq 0$. *Then $u$ is constant.*

If we try to extend the proofs presented in the $p = 2$ case, apparently the main obstruction is given by the following

**Remark 1.37.** *For general values of $p \neq 2$, if $f : N \to \mathbb{R}$ is convex and $u : M \to N$ is $p$-harmonic we do not have that $f \circ u$ satisfies $\Delta_p (f \circ u) \geq 0$. A family of counterexamples that use rotationally symmetric manifolds and maps was constructed by G. Veronelli in [53]. Before stating the main theorem of [53], we need to recall the next*

**Definition 1.38.** *An $m$-dimensional model manifold with warping functions $\sigma$ is a manifold $\mathbf{M}^m_\sigma$ diffeomorphic to $\mathbb{R}^m$ and endowed with a rotationally symmetric Riemannian metric. The model $\mathbf{M}^m_\sigma$ is realized as the quotient space $[0,+\infty) \times \mathbb{S}^{m-1}/\sim$ where $\sim$ identifies $\{0\} \times \mathbb{S}^{m-1}$ with the pole $0$ of the space, and the Riemannian metric has the expression*

$$g = dr \otimes dr + \sigma(r)^2 \, d\theta^2,$$

*where $d\theta^2$ denotes the standard metric of $\mathbb{S}^{m-1}$ and $\sigma : [0,+\infty) \to [0,+\infty)$ is a smooth function satisfying the following requirements:*

$$\text{(a) } \sigma(t) > 0, \ \forall t > 0; \qquad \text{(b) } \sigma'(0) = 1; \qquad \text{(c) } \sigma^{(2k)}(0) = 0,$$

*for every $k \geq 0$.*

**Remark 1.39.** *Spaceforms $\mathbf{M}^m(-c)$ of constant curvature $-c \leq 0$ are included in the picture. Indeed:*

*(1)* $\sigma(r) = r$ *corresponds to the Euclidean space* $\mathbb{R}^m$.

*(2)* $\sigma(r) = (c)^{-1/2} \sinh\left(c^{1/2} r\right)$ *corresponds to the hyperbolic space* $\mathbb{H}^m_{-c}$ *of constant curvature* $-c < 0$.

**Remark 1.40.** *Due to their explicit description and nice symmetries, model manifolds are typically considered as test spaces. Indeed, note that:*

*(i) The $r$-coordinate represents the distance from the pole $0$ of the model;*

*(ii) The Hessian of the distance function $r$ is given by*

$$\mathrm{Hess}(r) = \frac{\sigma'}{\sigma} \{\langle , \rangle - dr \otimes dr\}$$

*and, therefore, its Laplacian is*

$$\Delta r = (m-1) \frac{\sigma'}{\sigma};$$

*(iii) The volume element is given by* $\mathrm{dvol} = \sigma^{m-1}(r)\,dr d\theta$ *and, hence,*

$$\mathrm{area}\,(\partial B_R(0)) = c_m \sigma^{m-1}(r)\,;\ \mathrm{vol}\,(B_r(0)) = c_m \int_0^r \sigma^{m-1}(t)\,dt,$$

*where* $c_m$ *is the area of the* $(m-1)$*-dimensional unit sphere.*

*(iv) The sectional curvature in the radial direction* $\nabla r$ *is*

$$\mathrm{Sec}_{\mathrm{rad}} = -\frac{\sigma''}{\sigma}.$$

**Theorem 1.41.** *Let* $m \geq 2$ *and* $m+1 > p > \max\{m, 2\}$*. Consider the* $(m+1)$*-dimensional models* $\mathbf{M}_\sigma^{m+1}$ *and* $\mathbf{N}_\eta^{m+1}$ *where the warping functions* $\sigma$ *and* $\eta$ *are defined by, for* $r > 1$,

$$\sigma(r) = \left(r + a^{-\frac{1}{a-1}}\right)^a - a^{-\frac{a}{a-1}}, \qquad \eta(r) = \left(r + b^{\frac{1}{b-1}}\right)^b - b^{\frac{b}{b-1}},$$

*with*

$$a > \frac{1}{p-m} > 1$$

*and*

$$0 < b < 1.$$

*Then, there exists a* $C^2$*, rotationally symmetric p-harmonic map* $u(r,\theta) = (u(r), \theta) : \mathbf{M}_\sigma^{m+1} \to \mathbf{N}_\eta^{m+1}$ *and a sequence* $r_j \to +\infty$ *such that*

$$\Delta_p (f \circ u)(r_j) < 0,$$

*for every rotationally symmetric convex function* $f(r,\theta) = (h(r), \theta) : \mathbf{N}_\eta^{m+1} \to \mathbb{R}$ *satisfying* $f'(r) > 0$ *for every* $r > 0$.

**Problem 1.42.** *In the above example* $\mathbf{N}_\eta^{m+1}$ *has* $\mathrm{Sec}_{\mathrm{rad}} = -\eta''/\eta > 0$*. On the other hand, in applications the most interesting cases involve targets with* $\mathrm{Sec} \leq 0$*. It is an open question whether the good composition property fails even in these situations.*

**Proof (of Theorem 1.36).** As in the "linear" case, we suppose first that $N$ is Cartan-Hadamard.

Fix $x_0 \in M$ and let $y_0 = u(x_0) \in N$. Consider the "mixed" vector field

$$\begin{aligned} X &= |du|^{p-2}\, d\left(r_{y_0}^2 \circ u\right)^{\#} \\ &= d\left(r_{y_0}^2\right)(u) \circ \left(|du|^{p-2}\, du\right)^{\#}. \end{aligned}$$

where $r_{y_0} = d_N(\cdot, y_0)$ and $(\cdot)^{\#}$ denotes the musical isomorphism defined by $\langle \omega^{\#}, Y \rangle = \omega(Y)$. Then

$$\begin{aligned}\operatorname{div} X &= \operatorname{tr}_M \operatorname{Hess}\left(r_{y_0}^2\right)\left(|du|^{p-2}\, du, du\right) + d\left(r_{y_0}^2\right)(u)\,(\Delta_p u)\\ &= \operatorname{tr}_M \operatorname{Hess}\left(r_{y_0}^2\right)\left(|du|^{p-2}\, du, du\right).\end{aligned}$$

Since $r_{y_0}^2$ is strictly convex, $\operatorname{div} X \geq 0$ the equality holding if and only if $du = 0$, i.e., if and only if $u$ is constant. To conclude, apply the divergence theorem.

Now suppose that $N$ is a general complete manifold with $\operatorname{Sec}_N \leq 0$ and $u$ is homotopic to a constant. Define the equivariant vector field

$$X' = d\left(r_{y_0'}^2\right)(u') \circ \left(|du'|^{p-2}\, du'\right)^{\#}$$

where $r_{y_0'}(y') = d_{N'}(y', y_0')$, $u'$ is the lift of $u$ issuing from $x_0' \in P_M^{-1}(x_0)$ and $y_0' = u'(x_0')$. Then, $X'$ gives rise to a well defined vector field $Z$ on $M$ by

$$Z_x = X_{x'},$$

for any $x' \in P_M^{-1}(x)$. Arguing as above we conclude that $u'$, hence $u$, are constant. ■

**Remark 1.43.** *The idea of using vector fields instead of a mere composition of functions goes back to a nice paper by S. Kawai, [24].*

As an immediate consequence, we see that every map $u : M \to \mathbb{R}^k$ satisfying the $p$-harmonic system $\Delta_p u = 0$, for some $1 < p < +\infty$, must be constant. Actually, using the vector-structure of $\mathbb{R}^k$ we can mix the proofs of Theorems 1.13 and 1.36 to obtain the following much stronger conclusion. A proof, in the complete case, will be presented in Chapter 3, Theorem 3.3.

**Theorem 1.44.** *Let $M$ be a closed manifold. If $u, v : M \to \mathbb{R}^k$ are smooth solutions of*

$$\Delta_p u = \Delta_p v, \text{ on } M$$

*then* $u - v \equiv \text{const}.$

## 1.5.3 Concluding remarks

(A) In order to extend results from the linear to the non-linear setting the idea is to pass from composition of maps to the construction of suitable vector fields which are "modelled" on the composition. Moreover, the difficulties due to the presence of a non-trivial topology of the target are overcome by using equivariant theory and lifting to covering spaces. Obviously, this requires some condition on the homotopy class of the map.

(B) Theorem 1.36 is a comparison result for homotopic harmonic maps, one of which is the constant map. It is a classical result of P. Hartman, [16], in case $p = 2$, later extended to $p \neq 2$ by S.W. Wei, [56], that a nonconstant $p$-harmonic map $u : M \to N$ from a closed manifold $M$ into a closed target $N$ with $\mathrm{Sec}_N < 0$ is unique in its free homotopy class, unless it maps the domain manifold onto a closed geodesic of $N$.

(C) From a completely different perspective, in Theorem 1.36 one can avoid the topological assumption on the map up to imposing some curvature restriction on the source manifold $M$. For instance, a classical theorem by J. Eells and J.H. Sampson, [6], extended to $p \neq 2$ by S.W. Wei, [56], states that if $M$ and $N$ are closed manifolds with $\mathrm{Sec}_N \leq 0$, $\mathrm{Ric}_M \geq 0$ and $\mathrm{Ric}_M > 0$ at some point, then every $C^{1,\alpha}$ $p$-harmonic map $u : M \to N$ must be constant.

# Chapter 2

# Extending the divergence theorem to non-compact manifolds

From many viewpoints, we can think of parabolicity as a good and natural substitute of compactness. This is true, e.g., with respect to maximum principle properties of bounded functions and from the viewpoint of the divergence theorem for $L^q$-vector fields. The first aspect will be put as a definition of parabolicity and the second one will characterize parabolic manifolds. In between, a number of results will serve as a link between the two. In particular, some aspects concerning maximum principles on domains in parabolic manifolds will be analyzed in detail.

## 2.1 Some potential theory for the $p$-Laplacian

By its very definition, the $p$-Laplace operator and the corresponding $p$-potential theory, take into considerations the $p$-energy of functions. Indeed, it is canonically related to the $p$-energy functional

$$E_p(u) = \frac{1}{p}\int_M |\nabla u|^p \, d\text{vol}.$$

Recall from Chapter 1 that the $p$-Laplacian is the non-linear operator defined by

$$\Delta_p u = \text{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

We set the following definitions that extend in the obvious way those given for the Laplace-Beltrami operator.

**Definition 2.1.** *A function* $u \in W^{1,p}_{loc}(M) \cap C^0(M)$ *is said to be*

*(i)* *p-harmonic* *if* $\Delta_p u = 0$,

*(ii)* *p-subharmonic* *if* $\Delta_p u \geq 0$,

*(iii)* *p-superharmonic* *if* $\Delta_p u \leq 0$,

*the inequalities being intended in the sense of distributions.*

There are several ways to introduce the notion of a parabolic manifold for the $p$-Laplace operator. We choose to use the viewpoint of the Liouville-type property stated in the next

**Definition 2.2.** *A manifold* $M$ *is said to be* *p-parabolic,* $1 < p < +\infty$, *if every solution* $u \in W^{1,p}_{loc}(M) \cap C^0(M)$ *of the problem*

$$\begin{cases} \Delta_p u \geq 0 \text{ on } M, \\ \sup_M u < +\infty \end{cases}$$

*must be constant. A manifold which is not p-parabolic will be called* *p-hyperbolic.*

**Remark 2.3.** *Actually, since for every p-subharmonic function u the functions* $u_c = u + c$ *and* $u_+ = \max\{u, 0\}$ *are again p-subharmonic, in the above definition we can always assume that either* $0 \leq u \in L^\infty(M)$ *or* $u \leq 0$, *according to our purposes. Similarly, passing from* $u$ *to* $v = -u$ *yields* $\Delta_p v \leq 0$ *with* $\inf_M v > -\infty$. *Therefore,* $M$ *is p-parabolic if and only if every positive p-superharmonic function must be constant.*

**Remark 2.4** ($\varphi$-Laplacian extensions)**.** *Note that, formally, one can extend the above definition and the previous remark to the more general setting of the* $\varphi$*-Laplacian, i.e., the nonlinear elliptic operator* $L_\varphi$ *in divergence form introduced in Definition 1.11 of Chapter 1. If every solution of* $L_\varphi(u) \geq 0$ *with* $\sup_M u < +\infty$ *is constant, then the underlying manifold is called* $\varphi$*-parabolic. The terminology was introduced in [41].*

We are going to introduce several equivalent definitions that will play a role in the development of a global integration by parts theory. Before doing that, we point out that from the geometric viewpoint, parabolicity is reflected in the volume growth property of the manifold.

The following result, which extends classical theorems in the linear setting $p = 2$, was obtained independently by M. Troyanov, [49], I. Holopainen, [20], and M. Rigoli and A.G. Setti, [41]. The proof we shall present has a PDE slant: it is taken from [41] and represents a very special case of their general Lemma 1.1. It should be mentioned that, for $p = 2$, a similar argument was independently used by Q. Chen; see Lemma 2.3 in [4].

**Theorem 2.5.** *Let* $(M,g)$ *be a geodesically complete. If*

$$\int^{+\infty} \frac{dt}{\operatorname{area}\left(\partial B_t\left(o\right)\right)^{\frac{1}{p-1}}} = +\infty, \tag{2.1}$$

*for some origin* $o \in M$*, then* $M$ *is* $p$*-parabolic.*

**Proof.** Let $u$ be a $p$-subharmonic function. Without loss of generality we assume $u \le 0$. By contradiction, suppose that $u$ is non-constant on the ball $B_{R_0}(o)$ for some $R_0 > 1$. Let $0 \le \rho \in Lip_c(M)$ to be specified later. Applying the divergence theorem to the vector field

$$X = e^u \rho \left|\nabla u\right|^{p-2} \nabla u$$

we get

$$-\int_M \rho e^u \left|\nabla u\right|^p - \int_M e^u \left|\nabla u\right|^{p-2} \langle \nabla u, \nabla \rho \rangle \ge 0. \tag{2.2}$$

Now, having fixed $R > R_0$, and for every $\varepsilon > 0$, we choose $\rho = \rho_{R,\varepsilon}$ as follows

$$\rho(x) = \begin{cases} 1 & \text{on } B_R(o) \\ \dfrac{R+\varepsilon - r(x)}{\varepsilon} & \text{on } B_{R+\varepsilon}(o) \setminus B_R(o), \end{cases}$$

where we have set $r(x) = d(x,o)$. Then, from (2.2) we deduce

$$\varepsilon^{-1} \int_{B_{R+\varepsilon}(o)\setminus B_R(o)} e^u \left|\nabla u\right|^{p-2} \langle \nabla u, \nabla r \rangle \ge \int_{B_R(o)} e^u \left|\nabla u\right|^p.$$

Using the co-area formula and letting $\varepsilon \to 0^+$ in the LHS gives, for a.e. $R > R_0$,

$$\int_{\partial B_R(o)} e^u \left|\nabla u\right|^{p-2} \langle \nabla u, \nabla r \rangle \ge \int_{B_R(o)} e^u \left|\nabla u\right|^p. \tag{2.3}$$

Next note that, by the Cauchy-Schwarz and Hölder inequalities, and using that $u \le 0$, we have

$$\begin{aligned} \int_{\partial B_R(o)} e^u \left|\nabla u\right|^{p-2} \langle \nabla u, \nabla r \rangle &\le \int_{\partial B_R(o)} e^u \left|\nabla u\right|^{p-1} \\ &\le \left( \int_{\partial B_R(o)} e^u \right)^{\frac{1}{p}} \left( \int_{\partial B_R(o)} e^u \left|\nabla u\right|^p \right)^{\frac{p-1}{p}} \\ &\le \operatorname{area}\left(\partial B_R(o)\right)^{\frac{1}{p}} \left( \int_{\partial B_R(o)} e^u \left|\nabla u\right|^p \right)^{\frac{p-1}{p}}. \end{aligned}$$

Inserting into (2.3) we obtain

$$\int_{B_R(o)} e^u \left|\nabla u\right|^p \le \operatorname{area}\left(\partial B_R(o)\right)^{\frac{1}{p}} \left( \int_{\partial B_R(o)} e^u \left|\nabla u\right|^p \right)^{\frac{p-1}{p}},$$

which, in terms of the function

$$h(R) = \int_{B_R(o)} e^u |\nabla u|^p,$$

reads

$$\frac{h'(R)}{h(R)^{\frac{p}{p-1}}} \geq \frac{1}{\operatorname{area}(\partial B_R(o))^{\frac{1}{p-1}}}.$$

We integrate this latter on $[R_0, R]$ and let $R \to +\infty$ to conclude

$$\left(1 - \frac{1}{p}\right) \frac{1}{h(R_0)} \geq \int_{R_0}^{+\infty} \frac{1}{\operatorname{area}(\partial B_R(o))^{\frac{1}{p-1}}} = +\infty,$$

proving that

$$\int_{B_{R_0}(o)} e^u |\nabla u|^p = 0.$$

Therefore, $u$ is constant on $B_{R_0}(o)$. This contradicts our initial assumption. ■

**Remark 2.6** ($\varphi$-Laplacian extension)**.** *The same proof works without changes if we replace the p-Laplace operator with the $\varphi$-Laplacian $L_\varphi$. In this case, $p$ is just the structural constant associated to $\varphi$, so that $\varphi(t) \leq At^{p-1}$. Accordingly, if the volume growth condition (2.1) is satisfied, then every subsolution $L_\varphi(u) \geq 0$ with $\sup_M u < +\infty$ must be constant ,i.e., $M$ is $\varphi$-parabolic. This is the original formulation in [41].*

**Remark 2.7** (Parabolicity of models)**.** *On a model manifold $M^m_\sigma$ the volume growth condition (2.1) reads*

$$\int^{+\infty} \frac{dr}{\sigma^{m-1}(r)} = +\infty. \tag{2.4}$$

*It is easy to verify that this condition is in fact necessary and sufficient for the model to be parabolic. Indeed, if (2.4) does not hold, let $G(x) = \int_{r(x)}^{+\infty} \sigma^{1-m}(t)\, dt$. Then, $\Delta G(x) = 0$ on $M^m_\sigma \setminus \{o\}$ and $u(x) = \min\{1, \gamma(x)\}$ is a nonconstant, positive superharmonic function on $M^m_\sigma$, which is therefore nonparabolic. Indeed, it is readily verified that $G(x)$ is (proportional to) the Green's function with pole at $o$. On the other hand, if (2.4) holds, and we let $\gamma(x) = \int_1^{r(x)} \sigma^{1-m}(t)\, dt$ then $\Delta\gamma = 0$ on $M^M_\sigma \setminus \{o\}$ and $\gamma(x) \to +\infty$ as $x \to \infty$, and the parabolicity of $M^m_\sigma$ follows from the next very general result which usually goes under the name of Kha'sminski criterion. Versions of this result for non-linear operators appear in [37], and in the very recent [32]. The proof presented here is new.*

**Theorem 2.8.** *Let $q(x) \geq 0$ be a continuous function. Assume that there exists a function $\gamma > 0$ s.t. $\gamma \to +\infty$ and $\Delta\gamma \leq q(x)\gamma$ on $M \backslash K$, for some compact set $K \subset M$. If $0 \leq u \in L^{\infty}(M)$ solves $\Delta u \geq q(x) u$ then $u \equiv \mathrm{const}$.*

**Proof.** By contradiction assume that $u$ is non-constant. In particular, $u^* = \sup_M u > 0$. Since $\Delta u \geq 0$, by the maximum principle $u^*$ is not attained. Therefore, for any $\varepsilon > 0$ small enough $\Omega_\varepsilon = \{u - u^* + \varepsilon > 0\} \neq \varnothing$ is unbounded and $K \subset M \backslash \overline{\Omega}_\varepsilon$. Also, $\varepsilon > 0$ can be chosen in such way that $-u^* + \varepsilon < 0$. Let $u_\varepsilon = u - u^* + \varepsilon$ and note that $\Delta u_\varepsilon = \Delta u \geq q(x) u \geq q(x) u_\varepsilon$. Define

$$w = \frac{u_\varepsilon}{\gamma} \geq 0, \text{ on } \Omega_\varepsilon.$$

Clearly, $w = 0$ on $\partial\Omega_\varepsilon$ and $w(x) \to 0$ as $x \to \infty$. Since

$$\operatorname{div}\left(\gamma^2 \nabla w\right) = \gamma u_\varepsilon \left(\frac{\Delta u_\varepsilon}{u_\varepsilon} - \frac{\Delta\gamma}{\gamma}\right) \geq 0 \text{ on } \Omega_\varepsilon,$$

by the strong maximum principle, $w \equiv \mathrm{const}$ on every connected component of $\Omega_\varepsilon$. In particular, $u \geq u_\varepsilon = c\gamma$ for some constant $c > 0$, on an unbounded connected component of $\Omega_\varepsilon$. But this is impossible since $u$ is bounded. ■

### 2.1.1 Parabolicity & maximum principles

It is a well known consequence of the strong maximum principle, [22], that a $p$-superharmonic function $u \in C^0\left(\overline{\Omega}\right) \cap W^{1,p}(\Omega)$ on a bounded domain $\Omega \subset M$ attains its absolute minimum at some boundary point and, therefore,

$$\inf_{\Omega} u = \min_{\partial\Omega} u.$$

This is, in general, no longer true if $\Omega$ is unbounded and new phenomena, depending on global properties of the manifold, appear. For instance, consider the Euclidean space $\mathbb{R}^m$ realized as the rotationally symmetric manifold

$$\mathbb{R}^m = \left([0, +\infty) \times \mathbb{S}^{m-1}, \langle , \rangle = dr \otimes dr + r^2 ( , )_{\mathbb{S}^{m-1}}\right).$$

Fix $1 < p < m$ and consider the radial ($p$-Green) function $G : \mathbb{R}^m \backslash \{0\} \to \mathbb{R}$ defined by

$$G(x) = C \int_{r(x)}^{+\infty} \frac{dt}{t^{\frac{m-1}{p-1}}},$$

where $C > 0$ is chosen in such a way that $G(x) = 1$ on $\partial \mathbf{B}_1(0)$. Let $\Omega = \mathbb{R}^m \backslash \overline{\mathbf{B}}_1(0)$. Then, $G$ is $p$-harmonic in $\Omega$, it is positive (hence bounded from below) but

$$\min_{\partial\Omega} G = 1 > 0 = \inf_{\Omega} G.$$

Note that, since area $(\partial \mathbf{B}_r(0))^{-\frac{1}{p-1}} \approx r^{-\frac{m-1}{p-1}} \in L^1(+\infty)$, by Theorem 2.5 the manifold $\mathbb{R}^m$ is $p$-hyperbolic. This is in accordance with the following maximum principle characterization of $p$-parabolicity. The case $p = 2$ goes back to L.V. Ahlfors, whereas the general case $p \neq 2$ (together with the fact that exterior domains suffice) can be found in [34].

**Theorem 2.9.** *The Riemannian manifold $M$ is $p$-parabolic if and only if either of the following equivalent properties hold:*

*(a) For every smooth open set $\Omega \subset M$ with $\partial\Omega \neq \emptyset$ and for every $u \in C^0(\overline{\Omega}) \cap W^{1,p}_{loc}(\Omega)$ which is bounded from above and $p$-subharmonic in $\Omega$, it holds*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

*(b) There exists a smooth open set $D \subset\subset M$ such that for every $u \in C^0(M\backslash D) \cap W^{1,p}_{loc}(M\backslash\overline{D})$ which is bounded from above and it is $p$-subharmonic in $M\backslash\overline{D}$, it holds*

$$\sup_{M\backslash D} u = \max_{\partial D} u.$$

**Proof.** First, we show that properties (a) and (b) are equivalent. The implication (a)⇒(b) is trivial. Assume that (b) holds and suppose by contradiction that there exist a smooth open set $\Omega$ and a function $u$ as in (a) for which $\sup_{\partial\Omega} u < \sup_{\Omega} u$. From the standard comparison principle, it follows that $\Omega$ must be unbounded. Now, choose $0 < \varepsilon < \sup_{\Omega} u - \sup_{\partial\Omega} u$ sufficiently close to $\sup_{\Omega} u - \sup_{\partial\Omega} u$ so that $\overline{D} \cap \{u > \sup_{\partial\Omega} u + \varepsilon\} = \emptyset$. This is possible according to the strong maximum principle, because $\overline{D}$ is compact. Define $\tilde{u} \in C(M) \cap W^{1,p}_{loc}(M)$ by setting

$$\tilde{u}(x) = \max\{\sup_{\partial\Omega} u + \varepsilon, u(x)\}$$

and note that $\Delta_p \tilde{u} \geq 0$ on $M$. According to property (b),

$$\max_{\partial D} \tilde{u} = \sup_{M\backslash D} \tilde{u}.$$

However, since $\overline{D} \cap \{u > \sup_{\partial\Omega} u + \varepsilon\} = \emptyset$,

$$\max_{\partial D} \tilde{u} = \sup_{\partial\Omega} u + \varepsilon < \sup_{\Omega} u,$$

while

$$\sup_{M\backslash D} \tilde{u} = \sup_{\Omega} u.$$

The contradiction completes the proof of the implication (b)⇒(a).

Now we prove that condition (a) is equivalent to the $p$-parabolicity of the manifold. Assume that $M$ is $p$-hyperbolic, hence, there exists $u \in C(M) \cap W^{1,p}_{loc}(M)$ which is non-constant, bounded above and satisfies $\Delta_p u \geq 0$ weakly on $M$. Given $\gamma < \sup u$, then any connected component $\Omega$ of the set $\Omega_\gamma = \{u > \gamma\}$ is open, and $u$ is continuous and bounded above in $\overline{\Omega}$, satisfies $\Delta_p u \geq 0$ weakly in $\Omega$ and $\max_{\partial\Omega} u < \sup_\Omega u$. This proves that (a) is not satisfied.

Conversely, if (a) does not hold, then there exist a smooth open set $\Omega$ and a function $\psi \in C(\overline{\Omega}) \cap W^{1,p}_{loc}(\Omega)$ satisfying $\Delta_p \psi \geq 0$ and $\sup_\Omega \psi > \max_{\partial\Omega} \psi + 2\varepsilon$, for some $\varepsilon > 0$, then

$$\psi_\varepsilon = \begin{cases} \max\{\psi, \max_{\partial\Omega} \psi + \varepsilon\} & \text{in } \Omega, \\ \max_{\partial\Omega} \psi + \varepsilon & \text{in } M \backslash \Omega, \end{cases}$$

is a non-constant, bounded above, weak solution of $\Delta_p \psi_\varepsilon \geq 0$ on $M$. This proves that $M$ is $p$-hyperbolic. ∎

**Remark 2.10** (geometric implications). *We mention in passing that the Ahlfors maximum principle can be used e.g. to extend standard conclusions on minimal surfaces that rely on maximum principle considerations, from compact domains to unbounded subsets. By way of example, let $f : \Omega \subset M \to \mathbb{R}^m$ be a bounded minimal hypersurface with $\partial\Omega \neq \varnothing$ and $M$ parabolic. Then, $f(\Omega) \subseteq \operatorname{conv}(f(\partial\Omega))$. Indeed, let $L(x) = \sum a_j x^j$ be a supporting hyperplane for $f(\partial\Omega)$. Then $L(f) \leq 0$ on $\partial\Omega$. Since $\Delta L(f) = 0$ and $L(f)$ is bounded it follows from Ahlfors that $L(f) \leq 0$ on $\Omega$.*

**Remark 2.11** ($\varphi$-Laplacian extensions). *Inspection of the proof that p-parabolicity is equivalent to the validity of the Ahlfors maximum principle shows that only two properties of the p-Laplacian are used: (A) the p-Laplacian is not affected by adding a constant to the argument; (B) the maximum of p-subsolution is a p-subsolution. These two properties are shared by the $\varphi$-Laplacian; see Appendix C. The first one is obvious whereas the second one is a consequence of the fact that $L_\varphi$ is weakly elliptic and in divergent form. In conclusion, we have the following characterization that, apparently, was never observed before.*

**Theorem 2.12.** *A manifold $(M, g)$ is $\varphi$-parabolic if and only if the Ahlfors maximum principle holds on $M$.*

*This result has geometric implications on minimal graphs. For instance, let us point out the following*

**Corollary 2.13.** *Let $(M, g)$ be complete with $\operatorname{vol}(B_r) \leq r^2 \log r$. Let $\Gamma_u : \Omega \subset M \to M \times \mathbb{R}_{\geq 0}$ be a bounded minimal graph. If $\Gamma_u(\partial\Omega) \subset M \times \{0\}$ then $\Gamma_u \subset M \times \{0\}$.*

***Proof.*** *Indeed $M$ is $\varphi$-parabolic by Remark 2.6. Moreover, $u \geq 0$ is a bounded $\varphi$-harmonic function on $\Omega$ with $u = 0$ on $\partial\Omega$. Therefore, $u \equiv 0$.* ∎

### 2.1.2 Parabolicity & capacity

One of the most used characterizations of parabolicity involves the notion of capacity of condensers. This characterization has the great advantage that can be easily transplanted to more general metric spaces and gives immediate information on the robustness of the concept.

**Definition 2.14.** *A condenser is any couple $(K,\Omega)$ where $K$ is a compact set contained in the open set $\Omega$ of $M$. The p-capacity of the condenser $(K,\Omega)$ is defined by*

$$\mathrm{cap}_p(K,\Omega) = \inf\left\{\int_M |\nabla u|^p : u \in \mathcal{W}(K,\Omega)\right\},$$

*where $\mathcal{W}(K,\Omega)$ denotes the set of all $u \in Lip_c(\Omega)$ satisfying $u \geq 1$ on $K$. In case $\Omega = M$ we speak of the absolute p-capacity of $K$ and simply write* $\mathrm{cap}_p(K)$.

**Remark 2.15.** *From the definition it is clear that $\Omega \longmapsto \mathrm{cap}_p(K,\Omega)$ is a decreasing function whereas $K \longmapsto \mathrm{cap}_p(K,\Omega)$ is increasing. These monotonicity properties are used, for instance, to extend Definition 2.14 when either $K$ is replaced by a relatively compact open set $A$ or $\Omega$ is replaced by its closure $\overline{\Omega}$. One simply defines*

$$\mathrm{cap}_p(A,\Omega) = \sup_{K\subset A\ cpt} \mathrm{cap}_p(K,\Omega)$$

*and*

$$\mathrm{cap}_p\left(K,\overline{\Omega}\right) = \inf_{\overline{\Omega}\subset\Omega_1\ open} \mathrm{cap}_p(K,\Omega_1).$$

*From the first monotonicity property it also follows that*

$$\mathrm{cap}_p(K) = \lim_{\Omega\nearrow M} \mathrm{cap}_p(K,\Omega). \tag{2.5}$$

Using the direct method in the calculus of variations together with P. Tolksdorff regularity theory for $p$-harmonic functions, [48], it is not difficult to deduce the validity of the next

**Proposition 2.16.** *Let $\Omega_1 \subset\subset \Omega_2 \subset\subset M$ be precompact domains with sufficiently smooth boundaries. Then, there exists a unique function $u \in C^0\left(\overline{\Omega}_2\right) \cap C^{1,\alpha}(\Omega_2)$, which is called the p-equilibrium potential of $\left(\overline{\Omega}_1,\Omega_2\right)$, such that*

$$\mathrm{cap}_p\left(\overline{\Omega}_1,\Omega_2\right) = \int_M |\nabla u|^p.$$

*Furthermore, $0 \leq u \leq 1$ solves the boundary value problem*

$$\begin{cases} \Delta_p u = 0 & on\ \Omega_2 \\ u = 1 & on\ \partial\Omega_1 \\ u = 0 & on\ \partial\Omega_2. \end{cases}$$

In some sense, compact sets of zero $p$-capacity are invisible from the viewpoint of $p$-harmonic functions. For instance, if the singularities of a bounded $p$-harmonic function are confined in a certain compact region with vanishing capacity, then the original function extends $p$-harmonically even in these points; see e.g. [22]. In a $p$-parabolic manifold, all compact sets are negligible. The following result was originally proved by I. Holopainen, [18], using his non-linear Green function. We shall provide an alternative and very direct proof which is taken from [34].

**Theorem 2.17.** *A necessary and sufficient condition for $M$ to be $p$-parabolic is that, for every compact set $K \subset M$, $\mathrm{cap}_p(K) = 0$.*

**Proof.** Assume that, for every compact set $K$, $\mathrm{cap}_p(K) = 0$. By contradiction, suppose that there exists a positive, $p$-superharmonic function $u$. By translating and scaling, we may assume that

$$\text{(a)}\ \sup_M u > 1, \qquad \text{(b)}\ \inf_M u = 0. \tag{2.6}$$

Note that, by the strong maximum principle ([22] Theorem 7.12) $u$ is strictly positive on $M$. Next let $D$ be a relatively compact domain with smooth boundary contained in the superlevel set $\{u > 1\}$ and let $D_i$ be an exhaustion of $M$ consisting of relatively compact domains with smooth boundary such that $\overline{D} \subset\subset D_1$, and for every $i$ let $h_i$ be the solution of the the Dirichlet problem

$$\begin{cases} \Delta_p h_i = 0, \text{ on } D_i \setminus D, \\ h_i = 1, \text{ on } \partial D, \\ h_i = 0, \text{ on } \partial D_i. \end{cases}$$

By Tolksdorf regularity theory, [48], $h_i \in C^{1,\alpha}_{loc}\left(D_i \setminus \overline{D}\right)$. Furthermore, since $D$ and $D_i$ have smooth boundaries, applying Theorem 6.27 in [22] with $\theta$ any smooth extension of the piecewise function

$$\theta_0 = \begin{cases} 1, \text{ on } \partial D, \\ 0, \text{ on } \partial D_i, \end{cases}$$

we deduce that $h_i$ is continuous on $\overline{D}_i \setminus D$. By the strong maximum principle, we have $0 < h_i < 1$ in $D_i \setminus \overline{D}$ and using the comparison principle, [22, Lemma 3.18], we see that $\{h_i\}$ is an increasing sequence. Hence, by Dini's theorem, $\{h_i\}$ converges locally uniformly on $M \setminus D$ a function $h$ which is continuous on $M \setminus D$, $p$-harmonic on $M \setminus \overline{D}$ and satisfies $0 < h \leq 1$ on $M \setminus \overline{D}$ and $h = 1$ on $\partial D$. Again, $h \in C^0(M \setminus D) \cap C^{1,\alpha}_{loc}\left(M \setminus \bar{D}\right)$. Moreover, note that $h_i$ is the $p$-equilibrium potential of the condenser $(\overline{D}, D_i)$, therefore

$$\mathrm{cap}_p\left(\overline{D}, D_i\right) = \int |\nabla h_i|^p = \inf \int |\nabla \varphi|^p,$$

where the infimum is taken with respect to $\varphi \in Lip_c(D_i)$ such that $\varphi = 1$ on $\partial D$. Think of each $h_i$ extended to be zero off $D_i$. Therefore $\{\int_{M\setminus\overline{D}} |\nabla h_i|^p\}$ is decreasing and the sequence $\{h_i\} \subset W^{1,p}(\Omega)$ is bounded on every compact domain $\Omega$ of $M \setminus \overline{D}$. By the weak compactness theorem, see, e.g., Theorem 1.32 in [22], $h \in W^{1,p}(\Omega)$, and $\nabla h_i \to \nabla h$ weakly in $L^p(\Omega)$. In particular,

$$\int_\Omega |\nabla h|^p \le \liminf_{i\to+\infty} \int_{D_i\setminus D} |\nabla h_i|^p .$$

On the other hand, according to (2.5), $\lim_{i\to+\infty} \mathrm{cap}_p(\overline{D}, D_i) = \mathrm{cap}_p(\overline{D}) = 0$. Thus, letting $\Omega \nearrow M \setminus \overline{D}$ we conclude that

$$\int_{M\setminus\overline{D}} |\nabla h|^p = 0,$$

so that $h$ is constant, and since $h = 1$ on $\partial D$, $h \equiv 1$. Finally, since $u$ is $p$-superharmonic and $u \ge h_i$ on $\partial D \cup \partial D_i$, by the comparison principle, $u \ge h_i$ on $D_i \setminus D$, and letting $i \to \infty$ we conclude that $u \ge 1$ on $M$. This contradicts (2.6).

Suppose now the $M$ is $p$-parabolic in the PDEs sense of Definition 2.5. Given a relatively compact domain $D$, let $h_i$ and $h$ be the functions constructed above, and extend $h$ to be 1 in $D$, so that $h$ is continuous on $M$, bounded, and satisfies $\Delta_p h \le 0$ weakly on $M$. Thus $p$-parabolicity implies that $h$ is identically equal to 1. On the other hand, since the functions $h_i$ belong to $W^{1,p}_0(M)$, Lemma 1.33 in [22] shows that $\nabla h_i$ converges to $\nabla h$ weakly in $L^p(M)$. By Mazur's Lemma (see Lemma 1.29 in [22]) there exists a sequence $v_k$ of convex combinations of the $h_i$'s such that $\nabla v_k$ converges to $\nabla h$ strongly in $L^p$. Thus $v_k$ is continuous, compactly supported, identically equal to 1 on $\overline{D}$ (because so are all the $h_i$'s) and $\int_M |\nabla v_k|^p \to \int |\nabla h|^p = 0$, showing that $\mathrm{cap}_p(\overline{D}) = 0$, and $M$ is $p$-parabolic. ■

A nice feature of this characterization of parabolicity is the existence of special test functions. They resemble the usual radial cut-off functions for an increasing sequence of nested balls in a geodesically complete manifold; see Section 2.1.3 for a more precise discussion.

**Corollary 2.18.** *Suppose that $M$ is $p$-parabolic. Then, having fixed a smooth domain $\Omega \subset\subset M$, there exists an increasing sequence $\{\varphi_n\} \subset Lip_c(M)$ of functions satisfying the following requirements:*

*(a)* $0 \le \varphi_n \le 1$;

*(b)* $\varphi_n = 1$, *on* $\overline{\Omega}$;

*(c)* $\displaystyle\lim_{n\to+\infty} \int_M |\nabla\varphi_n|^p = 0$;

*(d)* $\lim_{n\to+\infty} \varphi_n(x) = 1$.

**Proof.** The construction of these test functions can be extrapolated from the above proof. Let us point out the main step for the sake of completeness. Choose a smooth, compact exhaustion $\{\Omega_n\}$ of $M$ with $\Omega_0 = \Omega$ and, for each $n$, define $\varphi_n$ as the $p$-equilibrium potential of the condenser $\left(\overline{\Omega}_0, \Omega_n\right)$ and extend it on all of $M$ as $\varphi_n = 0$ on $M \backslash \Omega_n$. By $p$-parabolicity and by the monotonicity properties of the $p$-capacity we conclude

$$\int_M |\nabla \varphi_n|^p = \mathrm{cap}_p\left(\overline{\Omega}_0, \Omega_n\right) \to \mathrm{cap}_p\left(\overline{\Omega}_0\right) = 0, \text{ as } n \to +\infty.$$

The fact that $\varphi_n \leq \varphi_{n+1}$ follows from the standard comparison principle. Finally, property (d) is a direct consequence of Theorem 2.9. Indeed, being the equilibrium potential of $\left(\overline{\Omega}, \Omega_n\right)$, each function $0 \leq \varphi_n \leq 1$ solves the corresponding problem

$$\begin{cases} \Delta_p \varphi_n = 0 & \text{on } \Omega_n, \\ \varphi_n = 1 & \text{on } \partial\Omega, \\ \varphi_n = 0 & \text{on } \partial\Omega_n. \end{cases}$$

By the Harnack principle, [22], the increasing sequence $\{\varphi_n\}$ converges to a function $\varphi \in C^0(M) \cap W^{1,p}_{loc}(M)$ which is $p$-harmonic on $M \smallsetminus \Omega$ and satisfies $0 \leq \varphi \leq 1$ and $\varphi|_{\partial\Omega} = 1$. Applying the Ahlfors maximum principle we deduce

$$\inf_{M \backslash \Omega} \varphi = 1 = \min_{\partial\Omega} \varphi,$$

proving that $\varphi \equiv 1$, as claimed. Finally, the fact that $\varphi_i$ is $Lip_c$ follows from the construction and Theorem 1 in [30]. ∎

### 2.1.3 Global divergence theorems for $L^q(q > 1)$-vector fields: the Kelvin-Nevanlinna-Royden criterion

From the viewpoint of a vector field $X \in L^{\frac{p}{p-1}}(M)$ a $p$-parabolic manifold $M$ has a negligible boundary at infinity and a global version of the divergence theorem holds. Actually, it happens that the validity of the divergence theorem for every chosen vector $L^{\frac{p}{p-1}}$ vector field completely characterizes the $p$-parabolicity of the underlying manifold. This is the content of the following important result which goes under the name of Kelvin-Nevanlinna-Royden criterion. In the linear setting $p = 2$ it was observed by T. Lyons and D. Sullivan, [31]. The non-linear version reported here is a contribution of V. Gol'dshtein and M. Troyanov, [9].

**Theorem 2.19.** *The Riemannian manifold $M$ is $p$-parabolic if and only if the following divergence theorem holds. Let $X$ be a vector field satisfying $|X| \in L^{\frac{p}{p-1}}$, $\operatorname{div} X \in L^1_{loc}(M)$ and $\operatorname{div} X$ has an integral, i.e., either*

$(\operatorname{div} X)_- \in L^1(M)$ *or* $(\operatorname{div} X)_+ \in L^1(M)$. *Then,*

$$\int_M \operatorname{div} X = 0.$$

**Proof.** Suppose that $M$ is $p$-parabolic. Without loss of generality, we assume that $(\operatorname{div} X)_+ \in L^1(M)$. Let $\{\varphi_n\}$ be the sequence of test functions constructed in Corollary 2.18. Since $\varphi_n X$ is compactly supported, the ordinary (weak) divergence theorem gives

$$0 = \int_M \operatorname{div}(\varphi_n X) = \int_M \langle \nabla\varphi_n, X\rangle + \int_M \varphi_n \operatorname{div} X.$$

Note that

$$\left|\int_M \langle \nabla\varphi_n, X\rangle\right| \leq \left(\int_M |\nabla\varphi_n|^p\right)^{\frac{1}{p}} \left(\int_M |X|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \to 0,$$

as $n \to +\infty$. On the other hand, decompose

$$\int_M \varphi_n \operatorname{div} X = \int_M \varphi_n (\operatorname{div} X)_+ - \int_M \varphi_n (\operatorname{div} X)_-,$$

so that

$$\int_M \varphi_n (\operatorname{div} X)_- = \int_M \langle \nabla\varphi_n, X\rangle + \int_M \varphi_n (\operatorname{div} X)_+.$$

Using property (d) in Corollary 2.18 and the monotone convergence theorem, and recalling that $(\operatorname{div} X)_+ \in L^1$, we deduce that

$$\int_M (\operatorname{div} X)_- = \int_M (\operatorname{div} X)_+ < +\infty$$

hence $(\operatorname{div} X)_- \in L^1(M)$ and

$$\int_M \operatorname{div} X = 0.$$

This completes the first part of the proof. The second half of the statement follows from a general result by M. Troyanov, [50], concerning the Poisson equation

$$\Delta_p u = h. \tag{2.7}$$

If $M$ is $p$-hyperbolic then, having fixed $0 \leq h \in C_c^\infty(M)$, $h \not\equiv 0$, there exists a solution $u \in C^{1,\alpha}_{loc}(M)$ of (2.7). Therefore, by defining $X = |\nabla u|^{p-2}\nabla u \in L^{p/(p-1)}$ we obtain a vector field that violates the global divergence theorem. ∎

As in the compact setting, closely related to the global integration by parts we have integral inequalities that can be obtained from the distributional condition $\operatorname{div} X \geq f$, with $f$ a locally integrable function.

**Proposition 2.20.** *Let* $(M, \langle\, ,\rangle)$ *be a* $p$*-parabolic Riemannian manifold,* $p > 1$*. Let* $X$ *be a vector field satisfying* $|X| \in L^{\frac{p}{p-1}}(M)$ *and*

$$\operatorname{div} X \geq f$$

*in the sense of distributions, for some* $f \in L^1_{loc}(M)$ *such that* $f_- \in L^1(M)$*. Then* $f \in L^1(M)$ *and*

$$\int_M f \leq 0.$$

*In particulare, if* $0 \leq f \in L^1_{\text{loc}}(M)$*, then*

$$f \equiv 0.$$

*Moreover, if* $\operatorname{div} X \geq 0$ *in the distributional sense then, for any* $0 \leq \alpha \in W^{1,p}(M) \cap L^\infty(M)$*, it holds*

$$\int_M \langle X, \nabla\alpha\rangle \leq 0.$$

**Proof.** Let $\{\Omega_j\}_{j=0}^\infty$ be an increasing sequence of precompact open sets with smooth boundaries such that $\Omega_j \nearrow M$ and let $\varphi_j$ be the cut-off functions constructed in Corollary 2.18 with $\Omega = \Omega_0$. Then,

$$\begin{aligned}\int_M \varphi_j f_+ &\leq (\operatorname{div} X, \varphi_j) + \int_M \varphi_j f_- \\ &= -\int_M \langle X, \nabla\varphi_j\rangle + \int_M \varphi_j f_- \\ &\leq \left(\int_M |X|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_M |\nabla\varphi_j|^p\right)^{\frac{1}{p}} + \int_M \varphi_j f_-.\end{aligned} \tag{2.8}$$

The first conclusion follows by letting $j \to +\infty$ and using the monotone convergence theorem in the on the left hand side and the dominated convergence theorem in the integrals on the right hand side. As for the second part of the statement, apply the above reasoning with the test functions $\eta = \varphi_j\alpha \in W^{1,p}_0(M) \cap L^\infty(M)$ and let $j \to +\infty$. ■

**Remark 2.21.** *Note that both the previous conclusions can be considered as a consequence of the formal application of the global divergence theorem and the formal identity* $\operatorname{div}(\alpha X) = \alpha \operatorname{div} X + \langle X, \nabla\alpha\rangle$*. Therefore, in the applications, we shall always use this more suggestive viewpoint.*

## 2.2 A notion of ∞-parabolicity

In their paper, [9], Gold'schtein-Troyanov proposed a way to include $p = +\infty$ in the above picture of $p$-parabolicity. We are grateful to

Marc Troyanov for having explained to us his viewpoint on the subject. Formally, let us express the $p$-harmonicity condition in the form

$$0 = \Delta_p u = (p-2)\,|\nabla u|^{p-2}\left\{\frac{1}{|\nabla u|^2}\mathrm{Hess}\,(u)\,(\nabla u, \nabla u) + \frac{1}{p-2}\Delta u\right\}$$

and letting $p \to +\infty$ we get

$$\mathrm{Hess}\,(u)\,(\nabla u, \nabla u) = 0.$$

Thus, one is led to think of this equation as a limit case of the $p$-harmonicity condition and put the following

**Definition 2.22.** *The operator $\Delta_\infty u = \mathrm{Hess}\,(u)\,(\nabla u, \nabla u)$ is called the* *$\infty$-Laplacian. We say that $u$ is $\infty$-harmonic ($\infty$-subharmonic or $\infty$-superharmonic) if $\Delta_\infty u = 0$ (respectively, $\geq 0$ or $\leq 0$).*

**Example 2.23.** *Every function $u : M \to \mathbb{R}$ satisfying $|\nabla u| \equiv \mathrm{const.}$ is $\infty$-harmonic. In particular, if $(M, g)$ is geodesically complete, there is a natural $\infty$-harmonic function of special interest: it is the distance function $r\,(x) = d\,(x, o)$ from a fixed reference origin. The fact that $|\nabla r| = 1$ is one of the equivalent formulations of the Gauss lemma.*

It is not clear whether or not it is possible to use the PDE viewpoint involving the $\infty$-Laplacian to formalize a useful notion of parabolicity. Existence and regularity theory for the $\infty$-Laplacian is a theory under construction (e.g. by M. Crandall, L.C. Evans and collaborators). Anyway, in a similar spirit, suppose to consider the $p$-energy functionals on $Lip_c\,(M)$ and observe that

$$\left\{\int_M |\nabla\varphi|^p\right\}^{1/p} \to \sup_M |\nabla\varphi|\,,\ \text{as } p \to +\infty. \tag{2.9}$$

This suggests to introduce a notion of absolute $\infty$-capacity as a "limit-case" situation of the ordinary $p$-capacity:

**Definition 2.24.** *Given a compact set $K \subset M$, let*

$$\mathrm{cap}_\infty\,(K) = \inf\,\|\nabla\varphi\|_\infty$$

*the infimum is taken with respect to all $\varphi \in Lip_c\,(M)$ satisfying $\varphi \geq 1$ on $K$.*

This leads naturally to formalizing a corresponding notion of parabolicity

**Definition 2.25.** *We say that the Riemannian manifold $(M, g)$ is $\infty$-parabolic if, for every compact set $K \subset M$, $\mathrm{cap}_\infty\,(K) = 0$.*

Recall that a nice feature of the capacitary characterization of $p$-parabolicity is the existence of special cut-off functions that, in turn, enable one to extend the divergence theorem to the non-compact realm. Similarly, we have the next

**Corollary 2.26.** *Let $M$ be an $\infty$-parabolic manifold. Having fixed a compact set $K$, there exists a sequence of functions $\varphi_n \in Lip_c(M)$ such that*

$$\text{(a) } \varphi_n \geq 0;\ \text{(b) } \varphi_n = 1, \text{ on } K;\ \text{(c) } \|\nabla\varphi_n\|_\infty \to 0, \text{ as } n \to +\infty.$$

**Proof.** The existence of a sequence of cut-off functions $f_n$ satisfying (b) and (c) follows immediately from the condition $\mathrm{cap}_\infty(K) = 0$. Now, take $\varphi_n = \max(f_n, 0)$ and note that $|\nabla\varphi_n| \leq |\nabla f_n|$. ■

**Remark 2.27.** *One may wonder if the fact that $M$ is $p$-parabolic for every $p >> 1$ implies that it is $\infty$-parabolic. This would be nice because it would support the intuition that $\infty$-parabolicity is a limit case of $p$-parabolicity. However it turns out that the implication does not hold. Indeed, if $M$ is a compact manifold, then it is $p$-parabolic for every $p$, and therefore so is $M \setminus \{point\}$. However the latter manifold is not geodesically complete and, by Theorem 2.29 below, not $\infty$-parabolic. Note that the reverse implication is also false, because there exist geodesically complete (hence $\infty$-parabolic, see Theorem 2.28) model manifolds which are not $p$-parabolic for every $p >> 1$. Simply take $M = \mathbf{M}^m_\sigma$ with $\sigma(r) = e^r$ when $r >> 1$.*

The class of $\infty$-parabolic manifolds is large enough to contain every geodesically complete manifold.

**Theorem 2.28.** *If $(M, g)$ is geodesically complete then it is $\infty$-parabolic.*

**Proof.** Fix a compact set $K \subset M$. Then $K$ is contained in a geodesic ball $B_{R_0}(o)$. Since $\mathrm{cap}_\infty(K)$ increases with $K$, we have

$$\mathrm{cap}_\infty(K) \leq \mathrm{cap}_\infty\left(\overline{B_{R_0}(o)}\right),$$

and it suffices to show that the RHS vanishes. For $n > R_0$, let $\varphi_n : [0, +\infty) \to \mathbb{R}$ be the piecewise linear function given by

$$\varphi_n(t) = \begin{cases} 1, & t \in [0, R_0], \\ 1 - \dfrac{t - R_0}{n - R_0}, & t \in [R_0, n], \\ 0, & t \in [n, +\infty). \end{cases} \tag{2.10}$$

Define the corresponding radial function

$$\varphi_n(x) = \varphi_n(r(x)) \in Lip_c(M).$$

Then

(a) $\varphi_n = 1$ on $\overline{B_{R_0}(o)}$, (b) $\varphi_n = 0$ off $B_n(o)$, (c) $\|\nabla\varphi_n\|_\infty \le 1/(n - R_0)$.

Therefore

$$\mathrm{cap}_\infty\left(\overline{B_{R_0}(o)}\right) \le \frac{1}{n - R_0} \to 0, \text{ as } n \to +\infty.$$

■

We are going to show that the opposite implication holds, namely, geodesic completeness and $\infty$-parabolicity are equivalent concepts. Later on, we shall show how one can deduce non-extendibility of $\infty$-parabolic manifolds using a Liouville theorem due to Yau. Although this is a weaker conclusion, we feel that the arguments are instructive.

**Theorem 2.29.** *Let $(M, g)$ be an $\infty$-parabolic manifold. Then $(M, g)$ is geodesically complete.*

**Proof.** Fix $o \in M$ and let $r(x) = d(x, o)$. According to the Hopf-Rinow theorem, it suffices to show that $r(x) \to +\infty$ as $x \to \infty$. This means that, having fixed $T > 0$ arbitrarily large, there exists a compact set $K_T \subset M$ such that $r(x) > T$ on $M \backslash K_T$. To this end, let $K$ be any compact neighbourhood of $o$. By the assumed $\infty$-parabolicity of $M$, for every $T$ there exists a function $\tilde{\varphi}_T \in Lip_c(M)$ such that $\tilde{\varphi}_T = 1$ on $K$ and $\|\nabla\tilde{\varphi}_T\|_\infty \le \frac{1}{2(T+1)}$. By using a Gaffney approximation procedure, [7, 15], based on local mollifiers, we produce a smooth, compactly supported function $\varphi_T$ which is 1 at $o$ and whose Lip-constant satisfies

$$\mathrm{Lip}(\varphi_T) \le \mathrm{Lip}(\tilde{\varphi}_T) + \frac{1}{2(T+1)} \le \frac{1}{(T+1)}.$$

Now let $K_T$ =supp$(\varphi_T)$. Then, for every $q \in M \backslash K_T$, and for every piecewise smooth path $\gamma : [0, 1] \to M$ from $\gamma(0) = o$ to $\gamma(1) = q$ we have

$$1 = |\varphi_T(o) - \varphi_T(q)| = \left|\int_0^1 \frac{d}{dt}\varphi_T(\gamma(t))\, dt\right| \le \int_0^1 \|\nabla\varphi_T\|_\infty |\dot{\gamma}(t)|\, dt \le \frac{1}{T+1}\ell(\gamma).$$

Since $\gamma$ was arbitrary, we conclude

$$1 \le \frac{1}{T+1} r(q).$$

■

## 2.2.1 Liouville & spectral consequences of $\infty$-parabolicity

A classical consequence of the existence of the cut-off functions constructed in Corollary 2.26 is the following Liouville type result by S.T. Yau.

**Theorem 2.30.** *Let $(M, g)$ be $\infty$-parabolic. If $u$ is a $p$-subharmonic function satisfying $u \in L^q(M)$, for some $0 < p-1 < q < +\infty$, then either $u < 0$ on $M$ or $u$ must be constant. In particular, every $p$-harmonic function $u \in L^q(M)$, for some $0 < p-1 < q < +\infty$, must be constant.*

**Remark 2.31** ($\varphi$-Laplacian extension)**.** *The same conclusions in Theorem 2.30 hold if we replace $p$-subharmonicity with $L_\varphi(u) \geq 0$ and $p$-harmonicity with $L_\varphi(u) = 0$, where $L_\varphi$ is the $\varphi$-Laplacian introduced in Definition 1.11 of Chapter 1. In this case, $p$ is the structural constant attached to the function $\varphi$ so that $\varphi(t) \leq At^{p-1}$. Furthermore, the vector field $X$ in the proof below has to be replaced by $Z = \eta^p u_\delta^{q-p+1} |\nabla u_\delta|^{-1} \varphi(|\nabla u_\delta|) \nabla u_\delta$.*

**Proof.** The idea of the proof is simply to obtain a Caccioppoli-type inequality for the function $u_+(x) = \max\{u(x), 0\} \in L^q(M)$. Let $\delta > 0$ be fixed and define $v_\delta = u_+ + \delta$. First of all, since $u$ is $p$-subharmonic then the same holds for $v_\delta$. Thus, applying the divergence theorem to the vector field

$$X = \eta^p v_\delta^{q-p+1} |\nabla v_\delta|^{p-2} \nabla v_\delta,$$

where $0 \leq \eta \in Lip_c(M)$ is a cut-off function to be specified later, we obtain

$$\begin{aligned}
0 &= \int_M \operatorname{div} X \\
&= (q-p+1) \int_M v_\delta^{q-p} \eta^p |\nabla v_\delta|^p \\
&+ p \int_M v_\delta^{q-p+1} \eta^{p-1} |\nabla v_\delta|^{p-2} \langle \nabla \eta, \nabla v_\delta \rangle \\
&+ \int_M \eta^p v_\delta^{q-p+1} \Delta_p v_\delta \\
&\geq (q-p+1) \int_M v_\delta^{q-p} \eta^p |\nabla v_\delta|^p + p \int_M v_\delta^{q-p+1} \eta^{p-1} |\nabla v_\delta|^{p-2} \langle \nabla \eta, \nabla v_\delta \rangle \\
&\geq (q-p+1) \int_M v_\delta^{q-p} \eta^p |\nabla v_\delta|^p - p \int_M v_\delta^{q-p+1} \eta^{p-1} |\nabla v_\delta|^{p-1} |\nabla \eta|,
\end{aligned}$$

i.e.

$$(q-p+1) \int_M v_\delta^{q-p} \eta^p |\nabla v_\delta|^p \leq p \int_M v_\delta^{q-p+1} \eta^{p-1} |\nabla v_\delta|^{p-1} |\nabla \eta|.$$

Whence, writing $q - p + 1 = (q-p)\frac{p-1}{p} + \frac{q}{p}$ and applying to the RHS the inequality

$$ab \leq \frac{(p-1)\varepsilon^{\frac{p}{p-1}}}{p} a^{\frac{p}{p-1}} + \frac{1}{p\varepsilon^p} b^p,$$

we get

$$\left\{q - p + 1 - \varepsilon^{\frac{p}{p-1}} (p-1)\right\} \int_M v_\delta^{q-p} \eta^p |\nabla v_\delta|^p \leq \frac{1}{\varepsilon^p} \int_M v_\delta^q |\nabla \eta|^p.$$

Clearly, since $q > p-1$, we can choose $\varepsilon > 0$ so small that the coefficient on LHS is strictly positive. On noting that $\nabla v_\delta = \nabla u_+$, letting $\delta \to 0^+$ and applying the monotone and dominated convergence theorems we obtain the required Caccioppoli inequality:

$$\left\{q-p+1-\varepsilon^{\frac{p}{p-1}}(p-1)\right\}\int_{\Omega_{\bar{n}}} u_+^{q-p}\left|\nabla u_+\right|^p \leq \frac{1}{\varepsilon^p}\int_M u_+^q\left|\nabla \eta\right|^p.$$

Now, fix an exhaustion $\Omega_n \nearrow M$ by relatively compact sets and, according to Corollary 2.26, for each $n > \bar{n}$, consider a sequence of cut-off functions $\eta_n \in Lip_c(M)$ satisfying $\eta_n \geq 0$ on $M$, $\eta_n = 1$ on $\Omega_{\bar{n}}$, and $\|\nabla \eta_n\|_\infty \to 0$. From the above inequality we obtain

$$\left\{q-p+1-\varepsilon^{\frac{p}{p-1}}(p-1)\right\}\int_{\Omega_{\bar{n}}} u_+^{q-p}\left|\nabla u_+\right|^p \leq \frac{1}{\varepsilon^p}\left\|\nabla \varphi_n\right\|_\infty^p\int_M u_+^q$$

and, letting $n \to +\infty$, we get

$$\int_{\Omega_{\bar{n}}} u_+^{q-p}\left|\nabla u_+\right|^p = 0.$$

Since this latter holds for every $\bar{n}$, we easily deduce that $u_+$ is constant. Thus either $u = \text{const} \geq 0$ or $u \leq 0$ and in the latter case, either $u < 0$ or $u \equiv 0$ by the strong maximum principle. ■

**Remark 2.32.** *In the case where $(M, g)$ is geodesically complete (i.e., $\infty$-parabolic), using the special cut-off functions defined in (2.10) we see that the conclusion of Theorem 2.30 can be obtained under the weaker condition*

$$\int_{B_R(o)} |u|^q = o(R^p), \text{ as } R \to +\infty.$$

*Actually, a similar use of these cut-off functions enables one to relax the $L^p$-integrability conditions in many other analytic and geometric results on a geodesically complete manifold, and Karp version of Gaffney theorem is no exception; see Theorem 2.38. So it is natural to inquire whether a better choice of cut-off functions is possible so to improve the uniform decay of their gradients and, therefore, to improve the corresponding geometric conclusions. The answer is essentially no; see Appendix A.*

As an application of Theorem 2.30 we show how to deduce that $\infty$-parabolic manifolds are not extendible. For the sake of completeness we recall the next

**Definition 2.33.** *A Riemannian manifold $(M, g)$ without boundary is said to be extendible if $M$ is isometric to a proper, open submanifold of a Riemannian manifold $(N, h)$.*

It is easy to show that every complete manifold is not extendible. However, the (flat) universal covering $M$ of $\mathbb{R}^2\backslash\{0\}$ is an example of a geodesically incomplete, not extendible manifold.

**Proposition 2.34.** *Let $(M,g)$ be an $m$-dimensional, $\infty$-parabolic manifold. Then $M$ is not extendible.*

**Proof.** By contradiction, suppose that there exists a Riemannian manifold $(N^m,h)$ such that $M^m \subset N^m$ as a proper, open subset. Take a point $x_0 \in \partial M$ and consider the geodesic ball $B_\varepsilon^N(x_0)$, $0 < \varepsilon << 1$. Let $G_{x_0}^N : B_\varepsilon^N(x_0) \to \mathbb{R}_{\geq 0}$ be the Green's kernel of $B_\varepsilon^N(x_0)$ with pole $x_0$, i.e., the minimal, positive fundamental solution of

$$\begin{cases} \Delta G(x) = -\delta_{x_0}(x) & \text{on } B_\varepsilon^N(x_0), \\ G = 0 & \text{on } \partial B_\varepsilon^N(x_0). \end{cases}$$

Then, it is known that,

$$G_{x_0}^N(x) \asymp \begin{cases} d(x,x_0)^{2-m} & \text{if } m > 2, \\ -\log d(x,x_0) & \text{if } m = 2, \end{cases} \quad \text{as } x \to x_0.$$

In particular,

$$G_{x_0}^N \in L^q\left(B_\varepsilon^N(x_0)\right),$$

for every

$$1 < q < \frac{m}{m-2}.$$

Now, extend $G_{x_0}^N$ to all of $N$ by setting

$$\bar{G}(x) = \max\left\{G_{x_0}^N(x) - \varepsilon, 0\right\}$$

and define $u \in C^0(M) \cap W_{loc}^{1,2}(M)$ as

$$u = \bar{G}\big|_M.$$

Clearly, $u$ is non-constant and $u \in L^q(M)$ for any $1 < q < m/(m-2)$. Moreover, $\Delta u \geq 0$ because $u$ is the maximum of solutions of the Laplace equation. This contradicts Theorem 2.30. ■

Yau's result has the following well known spectral consequence, first observed by R. Strichartz, [43].

**Corollary 2.35.** *Let $(M,g)$ be $\infty$-parabolic. Then, the Laplace-Beltrami operator is essentially self-adjoint.*

**Proof.** Indeed, it suffices to show that the equation

$$\Delta u = \lambda u, \text{ on } M$$

has no, non-zero solutions $u \in L^2(M)$ for any $\lambda > 0$. Thus, let $u$ be such a solution. Then $v = \max(u, 0) \in L^2(M)$ satisfies

$$\Delta v \geq \lambda v \geq 0, \text{ on } M.$$

By Theorem 2.30 we have $v \equiv \text{const}$, hence, $v \equiv 0$. This means that either $u \equiv \text{const} \geq 0$ or $u < 0$. In the second case, applying the reasoning to $-u$ we conclude that $u = \text{const}$. But then $0 = \Delta u = \lambda u$ forces $u = 0$. ■

### 2.2.2 Global divergence theorem for $L^1$-vector fields: Gaffney theorem

We observed before that a $p$-parabolic manifold has a negligible boundary at infinity from the viewpoint of $L^q$ vector fields, where $p$ is Hölder conjugate to $q$. Thus, a global divergence theorem for $L^q$ vector fields holds. Pushing forward even in this direction the formal similarities between $p$-parabolicity and $\infty$-parabolicity one expects that on an $\infty$-parabolic manifold there is a Stokes theorem for $L^1$ vector fields. The intuition is supported by the fact that what we needed to extend globally the divergence theorem is the existence of nice cut-off functions. In fact, we have the following result. As usual, we set $f_+ = \max(f, 0)$ and $f_- = \min(f, 0)$ so that $f = f_+ + f_-$

**Theorem 2.36.** *Let $M$ be $\infty$-parabolic. Then, for every vector field $X \in L^1(M)$ satisfying $\operatorname{div} X \in L^1(M)$ it holds*

$$\int_M \operatorname{div} X = 0.$$

**Proof.** Let $\Omega_n \nearrow M$ be a relatively compact exhaustion. For every $n$, choose a cut-off function $0 \leq \varphi_n \in Lip_c(M)$ as in Corollary 2.26 in such a way that $\varphi_n = 1$ on $\Omega_n$ and $\|\nabla \varphi_n\|_\infty < 1/n$. Then, we can apply the usual divergence theorem to the compactly supported vector field $\varphi_n X$ and obtain

$$0 = \int_M \operatorname{div}(\varphi_n X) = \int_M \langle \nabla \varphi_n, X \rangle + \int_M \varphi_n \operatorname{div} X. \tag{2.11}$$

Since $\varphi_n \to 1$ pointwise on $M$ and $\operatorname{div} X \in L^1(M)$, by dominated convergence we have

$$\int_M \varphi_n \operatorname{div} X \to \int_M \operatorname{div} X.$$

On the other hand

$$\left| \int_M \langle \nabla \varphi_n, X \rangle \right| \leq \|\nabla \varphi_n\|_\infty \|X\|_{L^1} \to 0, \text{ as } n \to +\infty,$$

and the desired conclusion follows by taking limits as $n \to +\infty$ in (2.11). ■

**Remark 2.37.** *Note that this result is just a reformulation in the language of* $\infty$*-parabolicity of a classical theorem by P.M. Gaffney, [7].*

# 2.3 Global divergence theorems: relaxing the $L^q$-assumptions

## 2.3.1 Karp version of the $L^1$-divergence theorem

Gaffney's result can be substantially improved. Indeed, the existence of radial cut-off functions with controlled decay of their gradients can be used to relax the $L^1$-integrability condition on the vector field $X$. Furthermore, in order to use the limit procedure of Theorem 2.36, we do not need the assumption $\operatorname{div} X \in L^1(M)$ in its full strength but the request that $\operatorname{div} X$ has a well defined integral clearly suffices. These simple observations led L. Karp to state the following

**Theorem 2.38.** *Let* $(M, g)$ *be complete (i.e.,*$\infty$*-parabolic) and let* $X \in L^1_{loc}(M)$ *be a vector field such that, for some origin* $o \in M$*,*

$$\int_{B_{2R}(o)\setminus B_R(o)} |X| = o(R), \text{ as } R \to +\infty.$$

*If either* $(\operatorname{div} X)_+ \in L^1(M)$ *or* $(\operatorname{div} X)_- \in L^1(M)$*, then*

$$\int_M \operatorname{div} X = 0.$$

**Proof.** As in (2.10) of Theorem 2.36, for each $R > 0$ let us consider a radial cut-off $\varphi_R(r(x)) \in Lip_c(M)$ satisfying

$$\begin{array}{ll} \text{(a) } 0 \le \varphi_R \le 1, \text{ on } M; & \text{(b) } \varphi_R = 1 \text{ on } B_R(o); \\ \text{(c) } \varphi_R = 0 \text{ off } B_{2R}(o); & \text{(d) } |\nabla \varphi_R| \le \frac{c}{R}. \end{array}$$

Then

$$0 = \int_M \operatorname{div}(\varphi_n X) = \int_M \langle \nabla \varphi_n, X \rangle + \int_M \varphi_n \operatorname{div} X.$$

Assume now that $(\operatorname{div} X)_+ \in L^1(M)$. Since $\varphi_n \to 1$ pointwise on $M$, by dominated convergence we have

$$\int_M \varphi_n (\operatorname{div} X)_+ \to \int_M (\operatorname{div} X)_+ .$$

On the other hand

$$\left| \int_M \langle \nabla \varphi_n, X \rangle \right| \le \frac{c}{R} \int_{B_{2R}(o)\setminus B_R(o)} |X| \to 0, \text{ as } n \to +\infty.$$

In particular, if we rewrite (2.11) in the form

$$\int_M \langle \nabla \varphi_n, X \rangle + \int_M \varphi_n (\operatorname{div} X)_+ = \int_M \varphi_n (\operatorname{div} X)_- ,$$

and, by taking limits as $n \to +\infty$, we deduce that $(\operatorname{div} X)_- \in L^1(M)$, and that

$$\int_M (\operatorname{div} X)_- = \int_M (\operatorname{div} X)_+ .$$

The desired conclusion easily follows. ■

### 2.3.2 An improved version of the $L^q (q > 1)$-divergence theorem

Since, in a precise sense, the KNR criterion represents an $L^p$-version of Gaffney-Stokes theorem, and since Gaffney's result on a geodesically complete (i.e., $\infty$-parabolic) manifold can be improved using special cut-off functions on balls, it is natural to ask whether or not it is possible to relax the $L^p$-condition on vector fields even on a $p$-parabolic manifold. This question was addressed and solved in case $p = 2$ by D. Valtorta e G. Veronelli in [55]. Their idea is to give a clever interpretation of sets, functions and quantities involved in Karp's formulation.

Suppose $(M, g)$ is complete (i.e., $\infty$-parabolic). Then, the distance function $r(x) = d(x, o)$ from an origin $o \in M$ enjoys the following properties:

$$\text{(a) } r(x) \to +\infty, \text{ as } x \to \infty; \text{ (b) } |\nabla r| = 1; \text{ (c) } \Delta_\infty r = 0 \text{ on } M \setminus \{o\}.$$

Obviously, each ball $B_R(o)$ can be considered as the (relatively compact) sublevel set

$$B_R(o) = \{x \in M : r(x) < R\}$$

and Karp's condition reads

$$\frac{1}{R} \|\nabla r\|_{L^\infty(\{R<r(x)<2R\})} \|X\|_{L^1(\{R<r(x)<2R\})} = o(1), \text{ as } R \to +\infty.$$

As for the proof, the needed cut-off functions are obtained from the piecewise linear functions

$$\varphi_R(t) = \begin{cases} 1, & t \in [0, R], \\ 1 - \dfrac{t-R}{R}, & t \in [R, 2R], \\ 0, & t \in [2R, +\infty), \end{cases}$$

by composing with the special function $r(x)$:

$$\varphi_R(x) = \varphi_R \circ r(x).$$

Thus, following step by step Karp's proof involving $L^1$ conditions we can see that the desired generalization to different $L^p$ conditions is reached once we are able to find a suitable replacement of the distance function in the $p$-parabolic setting. In case $p = 2$ this function is represented by the Evans potential of the (2-)parabolic manifold.

**Definition 2.39.** *Let $(M, g)$ be a Riemannian manifold. A p-Evans potential of $M$ is a function $e : M \to \mathbb{R}$ satisfying the following conditions:*

$$(a)\ e(x) \to +\infty,\ as\ x \to \infty;\ (b)\ \Delta_p e = 0\ on\ M \backslash \Omega,$$

*for some smooth domain $\Omega \subset\subset M$. In case $p = 2$ we simply speak of an Evans potential.*

**Remark 2.40.** *With this terminology, the distance function $r(x)$ is an $\infty$-Evans potential of the complete manifold $(M, g)$.*

It is not difficult to show that the existence of a $p$-Evans potential forces the underlying manifold to be $p$-parabolic. Actually we have the following more general result that goes under the name of Kha'sminskii criterion for $p$-parabolicity.

**Theorem 2.41.** *Let $(M, g)$ be a Riemannian manifold (not necessarily complete) and let $1 < p < +\infty$. Assume that there exists a function $u \in W^{1,p}_{loc}(M) \cap C^0(M)$ such that*

$$(a)\ u(x) \to +\infty,\ as\ x \to \infty;\ (b)\ \Delta_p u \le 0,\ on\ M \backslash \Omega$$

*for some $\Omega \subset\subset M$. Then $M$ is p-parabolic.*

In case $p = 2$, it is also known from classical works by L. Sario and M. Nakai, and Z. Kuramochi on surfaces, recently extended to every dimension e.g. by D. Valtorta in [51], that Evans potentials exist on every (2-)parabolic manifold.

**Theorem 2.42.** *Let $(M, g)$ be a parabolic Riemannian manifold. Then $M$ supports a smooth Evans potential $e : M \to \mathbb{R}$. Moreover*

$$\int_{\{e(x)<R\}} |\nabla e|^2 \le R.$$

With this preparation, it is now clear the validity of the following improvement of the $L^2$-Stokes theorem on 2-parabolic manifolds.

**Theorem 2.43.** *Let $(M, g)$ be 2-parabolic and let $X \in L^1_{loc}(M)$ be a vector field satisfying*

$$\int_{E(2R)\backslash E(R)} |X|^2 = o(R),\ as\ R \to +\infty,$$

*where*

$$E(R) = \{x \in M : e(x) < R\}$$

*and* $e : M \to \mathbb{R}$ *is an Evans potential of* $M$*. If either* $(\operatorname{div} X)_+$ *or* $(\operatorname{div} X)_-$ *are* $L^1$*-functions then*

$$\int_M \operatorname{div} X = 0.$$

**Problem 2.44.** *In order to consider this Theorem as a real improvement of the* $L^2$*-Stokes theorem we should be able to estimate the growth rate of* $E(R)$ *in terms of the geometry of* $M$*. This is an open question.*

**Problem 2.45.** *Formally, the result could be extended to every p provided we were able to prove the existence of p-Evans potentials on every p-parabolic manifold. This is an open question. An important indication of their existence comes from a recent work by D. Valtorta, [52], where it is proved that the Kha'sminskii criterion gives a necessary and sufficient condition for M to be p-parabolic. Thus, it remains to pass from p-superharmonic functions to p-harmonic functions.*

# Chapter 3

# Some applications of the global divergence theorems

## 3.1 The case of $L^q(q > 1)$-vector fields

### 3.1.1 Global comparison principles

The Stokes-oriented proofs we supplied in Chapter 1 both of the comparison results for real-valued solutions of PDEs and for the Liouville-type properties for solutions of systems of PDEs corresponding to $p$-harmonic maps all extend without changes to the parabolic setting, up to requiring suitable $L^p$-conditions on their energies.

More precisely, we have the following non-compact versions of Theorems 1.13, 1.36. The next result is from [21].

**Theorem 3.1.** *Assume that $(M, g)$ is p-parabolic, $p > 1$. Consider the $\varphi$-Laplace operator*

$$L_\varphi(u) = \operatorname{div}\left(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\right)$$

*corresponding to $\varphi \in C^0([0, +\infty)) \cap C^1((0, +\infty))$ satisfying the following structural conditions:*

$$(i)\ \varphi(0) = 0, \qquad (ii)\ \varphi(t) > 0\ \forall t > 0,$$
$$(iii)\ \varphi(t) \le At^{p-1}, \qquad (iv)\ \varphi(t)\ \text{is strictly increasing.}$$

*for some constants $A > 0, p > 1$. If $u, v \in W^{1,p}_{loc}(M) \cap C^0(M)$ are solutions of*

$$L_\varphi(u) \ge L_\varphi(v),\ \text{on } M$$

*and*

$$|\nabla u|, |\nabla v| \in L^p(M),$$

*then* $u - v \equiv \mathrm{const}$.

**Proof.** Keeping the notation of Theorem 1.13 , and using condition $(iii)$ on $\varphi$, we have that

$$X = \alpha(u-v)\left\{|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u - |\nabla v|^{-1}\varphi(|\nabla v|)\nabla v\right\} \in L^{\frac{p}{p-1}}(M)$$

and the global divergence theorem applies. ■

The following result can be found in [36].

**Theorem 3.2.** *Let $u : M \to N$ be a $p$-harmonic map homotopic to a constant. Suppose that $M$ is $p$-parabolic and that $N$ is complete with* $\mathrm{Sec}_N \le 0$*. Then $u$ is constant.*

**Proof.** Suppose $N$ is Cartan-Hadamard, the general case follows by using the equivariant argument.

Fix $x_0 \in M$, let $y_0 = u(x_0) \in N$ and consider the "mixed" vector field on $M$

$$\begin{aligned} X &= |du|^{p-2}\, d(\alpha \circ u) \\ &= d\alpha(u) \circ \left(|du|^{p-2}\, du\right). \end{aligned}$$

where $\alpha(x) = \beta(r_{y_0}(x))$ with $r_{y_0} = d_N(y, y_0)$ and $\beta : [0, +\infty) \to [0, +\infty)$ is a $C^2$-function satisfying $\beta' \ge 0$, $\beta'' \ge 0$ and

$$\beta(t) = \begin{cases} At^2 + B & \text{if } 0 \le t << 1 \\ t & \text{if } t \ge 1, \end{cases}$$

for suitable constants $A, B > 0$. Note that, by the Hessian comparison, $\mathrm{Hess}(\alpha) > 0$ in a small neighborhood of $y_0 \in N$. Note also that

$$|X| \le C\,|du|^{p-1} \in L^{\frac{p}{p-1}}(M),$$

and, by direct computations that use $p$-harmonicity of $u$ and the (weak) convexity of $\alpha$,

$$\begin{aligned} \mathrm{div} X &= |du|^{p-2}\,\mathrm{tr}_M\, \mathrm{Hess}_N(\alpha)|_u(du, du) + d\alpha(\Delta_p u) \\ &= |du|^{p-2}\,\mathrm{tr}_M\, \mathrm{Hess}_N(\alpha)|_u(du, du) \ge 0. \end{aligned}$$

Using the global divergence theorem we deduce

$$\mathrm{div} X \equiv 0,$$

showing that

$$|du(x)| = 0$$

on

$$\Omega = \left\{ x \in M : \mathrm{Hess}_N (\alpha)|_{u(x)} > 0 \right\}.$$

Since $\alpha$ is strictly convex near $y_0$, then $u$ must be constant in a neighborhood of $x_0$. Repeating the same construction with a different base point $x_0$ we conclude that $u$ is locally constant, hence a constant map. ■

Finally, we obtain a non-compact version of Theorem 1.44. The result is taken again from [21].

**Theorem 3.3.** *Let $(M, g)$ be a p-parabolic manifold. If $u, v : M \to \mathbb{R}^k$ are $C^2$ solutions of*

$$\Delta_p u = \Delta_p v, \text{ on } M$$

*with finite p-energy*

$$|du|^p, |dv|^p \in L^p(M),$$

*then* $u - v \equiv \text{const}$.

**Remark 3.4.** *We have already observed that the regularity imposed on u and v may appear quite unnatural in view of the fact that the PDEs involve the p-Laplacian, and so it is. In fact the proof we are presenting below is adapted from [55], where the authors in fact prove the theorem under the natural requirement that u and v are in $W^{1,p}_{loc} \cap C^0$. We specialize their proof to the case where u and v are $C^2$ for the sake of simplicity. The original proof can be carried out using weak instead of pointwise computations.*

**Proof.** We suppose that either $u$ or $v$ is non-constant, for otherwise there is nothing to prove. Fix $x_0 \in M$, let $C = u(x_0) - v(x_0) \in \mathbb{R}^n$ and let $\mathbf{r}(y) = |y - C|$ be the usual Euclidean distance from $C$. Note by replacing $v$ with $v - C$ we may assume that $C = 0$ and therefore $\mathbf{r}(y) = |y|$. For every fixed $A > 0$ consider the differentiable vector field $X_A$ on $M$ defined by

$$X_A(x) := \left[ d\alpha_A|_{(u-v)(x)} \circ \left( |du(x)|^{p-2} du(x) - |dv(x)|^{p-2} dv(x) \right) \right]^\sharp, \quad x \in M,$$

where $\alpha_A \in C^1(\mathbb{R}^n, \mathbb{R})$ is the smooth function

$$\alpha_A(y) := (A + \mathbf{r}(y)^2)^{1/2}$$

and $\sharp$ denotes is the musical isomorphism defined by $\langle \omega^\sharp, V \rangle = \omega(V)$ for all differential 1-forms $\omega$ and vector fields $V$. We observe that $X_A$ is well defined since there exists a canonical identification

$$T_{(u-v)(q)} \mathbb{R}^n \cong T_{u(q)} \mathbb{R}^n \cong T_{v(q)} \mathbb{R}^n \cong \mathbb{R}^n.$$

Since,

$$d\alpha_A(V) = \frac{d\mathbf{r}^2(V)}{2\sqrt{A + \mathbf{r}^2(y)}} = \frac{\langle y, V \rangle}{\sqrt{A + |y|^2}}$$

for every $V \in T_y\mathbb{R}^m$, we have

$$|X_A| \le \frac{|u-v|}{\sqrt{A+|u-v|^2}}\big||du(x)|^{p-2}du(x)-|dv(x)|^{p-2}dv(x)\big| \le (|du|^{p-1}+|dv|^{p-1}),$$

showing that $|X_A| \in L^{p/(p-1)}(M)$. The vector field $X_A$ is also weakly differentiable, with weak divergence given by

$$\begin{aligned}\operatorname{div} X_A &= d\alpha_A|_{(u-v)} \circ (\Delta_p u - \Delta_p v)\\ &\quad + \operatorname{tr}_M \operatorname{Hess}\alpha_A|_{(u-v)}\big(du-dv, |du|^{p-2}du - |dv|^{p-2}dv\big).\end{aligned}$$

The first term on the RHS vanishes by assumption. Moreover, for every $V, W \in T_y\mathbb{R}^m$,

$$\operatorname{Hess}\mathbf{r}^2(V,V) = 2\langle V, W\rangle,$$

and therefore

$$\begin{aligned}\operatorname{Hess}\alpha_A(V,W) &= \frac{\operatorname{Hess}\mathbf{r}^2|(V,W)}{2[A+\mathbf{r}^2]^{1/2}} - \frac{d\mathbf{r}^2 \otimes d\mathbf{r}^2(V,W)}{4[A+d\mathbf{r}^2]^{3/2}}\\ &= \frac{\langle V,W\rangle}{[A+|y|^2]^{1/2}} - \frac{\langle y,V\rangle\langle y,W\rangle}{[A+|y|^2]^{3/2}}.\end{aligned}$$

Thus, if $E_i$ is a local o.n. basis at $x$ we get

$$\begin{aligned}\operatorname{div} X_A &= \sum_i \frac{\langle (du-dv)(E_i), (|du|^{p-2}du - |dv|^{p-2}dv)(E_i)\rangle}{[A+|u-v|^2]^{1/2}}\\ &\quad -\sum_i \frac{\langle u-v, (du-dv)(E_i)\rangle\langle u-v, (|du|^{p-2}du-|dv|^{p-2}dv)(E_i)\rangle}{[A+|u-v|^2]^{3/2}}\\ &\ge \frac{\langle (du-dv), (|du|^{p-2}du-|dv|^{p-2}dv)_{HS}\rangle}{[A+|u-v|^2]^{1/2}}\\ &\quad -\frac{|u-v|^2|du-dv|\big||du|^{p-2}du-|dv|^{p-2}dv\big|}{[A+|u-v|^2]^{3/2}}\\ &\ge \frac{h(du,dv)}{[A+|u-v|^2]^{1/2}} - \frac{2|u-v|^2(|du|^p+|dv|^p)}{[A+|u-v|^2]^{3/2}} =: f_A,\end{aligned} \tag{3.1}$$

where

$$h(du,dv) = \langle (du-dv), (|du|^{p-2}du-|dv|^{p-2}dv)\rangle_{HS},$$

$\langle\, ,\rangle_{HS}$ is the usual Hilbert-Schmidt metric in $T^*M \otimes R^n$, and in the last inequality we have used Young's inequality $ab \le p^{-1}a^p + (p-1)p^{-1}b^{p/(p-1)}$.

Since $\operatorname{div} X_A \ge f_A$ weakly on $M$, and, by Lemma 1.14,

$$f_A \ge -\frac{2(|du|^p+|dv|^p)}{\sqrt{A}} \in L^1(M), \tag{3.2}$$

it follows from Proposition 2.20 that

$$0 \geq \int f_A.$$

Now fix $\lambda > 0$ and consider the sets

$$M_\lambda = \{x : |u(x) - v(x)| < \lambda\}, \quad M^\lambda = M \setminus M_\lambda,$$

and, recalling the definition of $f_A$ write

$$0 \geq \int f_A = \int_{M^\lambda} f_A + \int_{M_\lambda} \frac{h(du, dv)}{[A + |u - v|^2]^{1/2}} - \int_{M_\lambda} \frac{2|u - v|^2(|du|^p + |dv|^p)}{[A + |u - v|^2]^{3/2}}. \tag{3.3}$$

According to (3.2) we have

$$\int_{M^\lambda} f_A \geq -\frac{2}{\sqrt{A + \lambda^2}} \int_{M^\lambda} (|du|^p + |dv|^p). \tag{3.4}$$

Also,

$$\int_{M_\lambda} \frac{h(du, dv)}{[A + |u - v|^2]^{1/2}} \geq \frac{1}{\sqrt{A + \lambda^2}} \int_{M_\lambda} h(du, dv). \tag{3.5}$$

Moreover, since the real valued function $\lambda \to 2\lambda^2/(A+\lambda^2)^{3/2}$ it is increasing in the interval $[0, \sqrt{2A}]$ and attains its absolute maximum at $\lambda = \sqrt{2A}$ we have

$$\int_{M_\lambda} \frac{2|u - v|^2(|du|^p + |dv|^p)}{[A + |u - v|^2]^{3/2}} \leq \frac{2\lambda^2}{[A + \lambda^2]^{3/2}} \int_{M_\lambda} (|du|^p + |dv|^p), \tag{3.6}$$

provided $\lambda \leq \sqrt{2A}$. Inserting the above inequalities into (3.3) and rearranging gives the inequality

$$\int_{M_\lambda} h(du, dv) \leq 2 \int_{M^\lambda} (|du|^p + |dv|^p) + \frac{2\lambda^2}{A + \lambda^2} \int_{M_\lambda} (|du|^p + |dv|^p), \tag{3.7}$$

valid for all $\lambda \leq \sqrt{2A}$. To conclude, let first $A \to \infty$, so that the second summand on the right hand side tends to zero. Next, let $\lambda \to \infty$. Since $|du|^p + |dv|^p \in L^1(M)$, the first integral on the right hand side tends to zero as $\lambda \to \infty$ and since $h(dv, du) \geq 0$ by Lemma 1.14, by monotone convergence we obtain that

$$\int_M h(du, dv) = 0.$$

It follows that $h(du, dv) = 0$, and therefore, again by Lemma 1.14, $du = dv$ whence $u - v \equiv u(x_0) - v(x_0) = 0$ on $M$. ∎

### 3.1.2 Concluding remarks on comparison & Liouville theory

(A) The comparison theory for homotopic $p$-harmonic maps with finite $p$-energy starts with the impressive work by R. Schoen and S.T. Yau in the case $p = 2$; see [45] and [46]. The first step forward in the general case $p > 2$ is done in [36] where the case of a $p$-harmonic map homotopic to a constant is considered. Intermediate results are then obtained in [21]. The complete analogue of the Schoen-Yau comparison theory for $p$-harmonic maps in the same homotopy class can be found in the very recent paper [54] by G. Veronelli.

(B) From a different perspective, one can investigate Liouville-type conclusions for $p$-harmonic maps with finite $p$-energy when the topological assumption on the map is replaced by a curvature condition on the domain manifold. In this direction, we mention e.g. the seminal paper [44] by R. Schoen and S.T. Yau in the case $p = 2$, later extended by N. Nakauchi, [33], and K. Takegoshi, [47], to $p \neq 2$. In all these papers the domain manifold has non-negative Ricci curvature. Actually, a certain amount of negative Ricci curvature can be allowed as shown in [35] and [38] for $p = 2$, and in [40] for $p > 2$. This Liouville-type theorems have applications to the homtopy class of general maps with finite $p$-energy into compact manifolds of non-positive curvature.

## 3.2 The case of $L^1$-vector fields

### 3.2.1 Liouville theorem for harmonic functions of sublinear growth

We began Chapter 1 by recalling that on a compact manifold, (sub)harmonic functions are necessarily constant. S.Y. Cheng and S.T. Yau, [58], [5], extended this conclusion to complete, non-compact manifolds by showing that, if $Ric \geq 0$ then harmonic functions with sublinear growth are constant. Using the Karp divergence theorem we can obtain the following $Ric$-quantitative version of Cheng-Yau result where a possible negativity of the Ricci curvature is balanced by a suitable integral control of the function.

**Proposition 3.5.** *Let $(M, g)$ be a complete manifold satisfying*

$$Ric\,(x) \geq -\frac{K^2}{(1 + r\,(x))^{2\alpha}},$$

*with* $r(x) = d(x,o)$, $\alpha, K \geq 0$. *Let* $u$ *be a harmonic function satisfying*

$$\int_{B_{2R}\setminus B_R} |u| = O\left(R^{\beta}\right), \text{ as } R \to +\infty \tag{3.8}$$

*and*

$$\sup_{\partial B_R(o)} |u| = o\left(R^{\gamma}\right), \text{ as } R \to +\infty,$$

*where* $\beta, \gamma \geq 0$ *are subjected to the restriction*

$$\beta + \gamma - \min\{\alpha, 1\} - 1 \leq 0. \tag{3.9}$$

*Then* $u$ *is constant. If* $K = 0$ *we can choose* $\alpha = 1$ *in* (3.9), *and if, additionally,* $\gamma = 1$ *the conclusion holds without any extra assumptions on* $u$ *(thus recovering the original statement by Cheng-Yau).*

**Remark 3.6.** *Note that if* $0 \leq u \in L^1(M)$, *then the result follows immediately from the stochastic completeness of* $M$, *regardless of any growth condition on* $u$. *We also note that it follows from work of P. Li and R. Schoen, [28], that if* $Ric \geq 0$, *and* $u \in L^p(M)$, *for some* $p \in (0, +\infty]$ *then* $u$ *is necessarily zero.*

We shall use the following classical result by S.Y. Cheng and S.T. Yau which is known in the literature as the gradient estimates for harmonic functions.

**Theorem 3.7.** *Let* $(M, g)$ *be a Riemannian manifold such that* $Ric \geq -A$ *on the relatively compact geodesic ball* $B_{2R}(\bar{x})$, *for some* $A \geq 0$. *If* $u > 0$ *is harmonic in* $B_{2R}(\bar{x})$ *then, there exists a universal constant* $C > 0$ *such that*

$$\frac{|\nabla u|}{u} \leq C\left(\frac{1}{R} + \sqrt{A}\right), \text{ on } B_R(\bar{x}).$$

**Proof.** We consider first the case $K = 0$. Given $R > 0$, applying the Cheng and Yau gradient estimate to the function $u + \sup_{B_{2R}(o)} |u| + 1$ on the ball $B_R(o)$, and using the assumed growth condition on $u$ we obtain

$$|\nabla u|(x) \lesssim \frac{1}{R}\left(u(x) + \sup_{B_{2R}(o)} |u| + 1\right) = o\left(R^{\gamma-1}\right),$$

valid for every $x \in B_R(o)$. In the case where $\gamma = 1$, letting $R \to 0$, this shows that for every fixed $x$, $|\nabla u(x)| = 0$. Therefore $u$ is constant in this case, and we recover Cheng and Yau's original result.

If $\gamma > 1$, we consider the vector field $X = u\nabla u$. Since $\operatorname{div} X = u\Delta u + |\nabla u|^2 = |\nabla u|^2$, $(\operatorname{div} X)_- = 0$ is clearly integrable. Moreover, using assumption (3.8) shows that $X$ satisfies

$$\begin{aligned}\frac{1}{R}\int_{B_{2R}\setminus B_R}|X| &= \frac{1}{R}\int_{B_{2R}\setminus B_R}|\nabla u|\,|u| \\ &\le \frac{1}{R}\sup_{B_{2R}\setminus B_R}|\nabla u|\int_{B_{2R}\setminus B_R}|u| \\ &= o\left(R^{\beta+\gamma-2}\right) \\ &= o\,(1)\,,\end{aligned}$$

provided $\beta+\gamma-2 \le 0$. Thus, by Karp's theorem (Theorem 2.38) we conclude that

$$0 = \int_M \operatorname{div} X = \int_M |\nabla u|^2\,,$$

and $u$ is constant.

The case where $K > 0$ is similar. The starting point is again the Cheng and Yau gradient estimate. Given $x$ with $r(x) = R$, we apply the gradient estimate on $B_{R/4}(x)$ to $u + \sup_{B_{2R}}(o)|u| + 1$. Since the Ricci curvature satisfies

$$Ric \ge -\frac{K^2}{(1+R/2)^{2\alpha}} \quad \text{on } B_{R/2}(x),$$

using the growth assumption on $u$ we deduce that

$$|\nabla u|\,(x) \lesssim \left(\frac{1}{R}+\frac{K}{(1+R)^{\alpha}}\right)\left(u\,(x)+\sup_{B_{2R}(o)}|u|\right) = o\left(R^{\gamma-1}\right)+o\left(R^{\gamma-\alpha}\right).$$

Now the proof proceeds as in the case $K = 0$, noting that

$$\frac{1}{R}\int_{B_{2R}\setminus B_R}|X| = o\left(R^{\beta+\gamma-\min\{\alpha,1\}-1}\right) = o\,(1)\,,$$

provided $\beta+\gamma-\min\{\alpha,1\}-1 \le 0$. ∎

### 3.2.2 Lower volume estimates & rigidity for the sharp Sobolev constant

We say that a Riemannian manifold $(M,g)$ of dimension $\dim M = m > p \ge 1$ supports an Euclidean-type Sobolev inequality if there exists a constant $C_M > 0$ such that, for every $u \in C_c^\infty(M)$,

$$\left(\int_M |u|^{p^*}\,d\mathrm{vol}\right)^{\frac{1}{p^*}} \le C_M\left(\int_M |\nabla u|^p\,d\mathrm{vol}\right)^{\frac{1}{p}}, \tag{3.10}$$

where

$$p^* = \frac{mp}{m-p}.$$

If $M$ has at most an Euclidean volume growth, i.e.,

$$\mathrm{vol}\left(B_R(o)\right) \le CR^m,$$

for some origin $o \in M$ and for some $C > 0$, then it is not difficult to show by standard density arguments that (3.10) can be expressed in the equivalent form

$$C_M^{-p} \le \inf_{u \in \Lambda} \int_M |\nabla u|^p \, d\mathrm{vol},$$

where

$$\Lambda = \left\{ u \in L^{p^*}(M) : |\nabla u| \in L^p \text{ and } \int_M |u|^{p^*} \, d\mathrm{vol}_M = 1 \right\}.$$

Indeed, the main point is to show that if $u \in L^{p^*}(M)$ and $|\nabla u| \in L^p(M)$ then there exists a sequence of compactly supported functions $u_n \in L^{p^*}$ with $|\nabla u_n| \in L^p$ such that

$$u_n \to u \text{ in } L^{p^*} \quad \text{ and } \nabla u_n \to \nabla u \text{ in } L^p.$$

Once this is done, usual regularization techniques allow the approximation using sequences in $C_c^\infty$. Now, for every $R > 0$, let $\rho_R$ be a $C^\infty$ cut-off function such that $\rho_R = 1$ on $B_R(o)$, $\rho = 0$ off $B_{2R}(o)$ and $|\nabla \rho_R| \le CR^{-1}\chi_{B_{2R}(o) \setminus B_R(o)}$, for some $C > 1$. Then $u_R \to u$ in $L^{p^*}$, by dominated convergence, while

$$\int_M |\nabla (u - u_R)|^p \le 2^{p-1} \Big( \int_M |\nabla u|^p + \int_M |u|^p |\nabla \rho_R|^p \Big),$$

and the first integral converges to zero, again by dominated convergence, while, using Hölder's inequality and the volume growth assumption, we have

$$\int_M |u|^p |\nabla \rho_R|^p \le C^p \Big( \int_{B_{2R}(o) \setminus B_R(o)} |u|^{p^*} \Big)^{\frac{m-p}{m}} \left( R^{-m} \mathrm{vol} B_{2R}(o) \right)^{p/m}$$

and the right hand side tends to zero by another application of dominated convergence.

The validity of (3.10), as well as the best value of the Sobolev constant $C_M$, have intriguing and deep connections with the geometry of the underlying manifold, many of which are discussed in the excellent lecture notes [17]. See also [27] for a survey in the more abstract perspective of Markov diffusion processes, and [13] for the relevance of (3.10) in the $L^{p,q}$-cohomology theory. For instance, we note that a complete manifold with non-negative Ricci curvature (but, in fact, a certain amount of negative curvature is allowed) and supporting an Euclidean-type Sobolev inequality is necessarily connected at infinity. This fact can be proved using the potential theoretic arguments we developed in Chapter 2; see [35], [34].

It is known (see e.g. Proposition 4.2 in [17]) that

$$C_M \geq K(m,p), \tag{3.11}$$

where $K(m,p)$ is the best constant in the corresponding Sobolev inequality of $\mathbb{R}^m$. It was discovered in the influential paper by M. Ledoux, [26], that for complete manifolds of non-negative Ricci curvature, the equality in (3.11) forces $M$ to be isometric to $\mathbb{R}^m$. This important rigidity result has been generalized by C. Xia, [57], by showing that, if $C_M$ is sufficiently close to $K(m,p)$, then $M$ is diffeomorphic to $\mathbb{R}^m$. The key ingredient in the Ledoux-Xia argument is a sharp lower estimate for the growth of geodesic balls which depends explicitly on the Sobolev constant. Actually, it is known, [2], [1], that the validity of (3.10) implies that there exists a (small) constant $\gamma = \gamma(m,p,C_M) > 0$ (depending continuously on $C_M$) such that

$$\mathrm{vol}(B_R) \geq \gamma \mathrm{vol}(\mathbf{B}_R), \tag{3.12}$$

where $\mathbf{B}_R$ denotes the Euclidean ball. However, to obtain the desired rigidity one needs to get a sharp value for $\gamma$, and this improvement, in the proofs supplied by Ledoux and Xia, heavily relies of the curvature assumption. According to Ledoux this should be just a limit imposed by the techniques involved. In fact

**Conjecture 3.8** (Ledoux)**.** *Let $(M,g)$ be a complete Riemannian manifold enjoying the Euclidean Sobolev inequality (3.10) with sharp Euclidean Sobolev constant $C_M = K(m,p)$. Then*

$$\mathrm{vol}(B_R) \geq \mathrm{vol}(\mathbf{B}_R).$$

The Ricci curvature assumption in Ledoux-Xia results was substantially relaxed in the recent paper [39] where, furtheremore, it is introduced a new method based on Karp integration and comparison arguments involving the Laplacian of the distance function. We are going to exemplify this method in the simplified form needed to get the original result by Ledoux. It is important to observe that, in a recent paper [3], G. Carron has been able to replace the curvature condition with an asymptotic volume growth assumption thus strengthening and extending the main results in [39]. His proof relies on a clever elaboration of the original arguments by Ledoux.

**Theorem 3.9.** *Let $(M,g)$ be a complete, m-dimensional Riemannian manifold, $m > p > 1$. Assume that $\mathrm{Ric}_M \geq 0$ and that the Euclidean-type Sobolev inequality (3.10) holds on $M$. Then*

$$\mathrm{area}(B_R) = \mathrm{area}(\mathbf{B}_R), \ \forall R \geq 0, \tag{3.13}$$

*where $\mathbf{B}_R$ denotes the ball of radius $R > 0$ in the standard Euclidean space $\mathbb{R}^m$. In particular, $M$ is isometric to $\mathbb{R}^m$.*

As we shall see momentarily, the proof of Theorem 3.9 resembles the proof of the celebrated eigenvalue comparison by S.Y. Cheng. Thus, the main ingredients are:

(A) a suitable PDE satisfied by the extremal function for the Euclidean case

(B) The comparison geometry for the Ricci tensor

(C) A suitable integration by parts on the whole manifold, which is provided by Karp's theorem (in Cheng result, one compares solutions of PDEs on balls, so that the usual Stokes theorem suffices).

As for points (A) and (B) we recall the following very important facts.

**Lemma 3.10** (extremal functions). *On the Euclidean space* $\mathbb{R}^m$*, the equality in (3.10) with the best constant* $C_{\mathbb{R}^m} = K(m,p)$*, is realized by the (radial) Bliss-Aubin-Talenti functions* $\phi_\lambda(x) = \varphi_\lambda(|x|)$ *for every* $\lambda > 0$*, where* $|x|$ *is the Euclidean norm of* $x$ *and* $\varphi_\lambda(t)$ *are the real-valued functions defined as*

$$\varphi_\lambda(t) = \frac{\beta(m,p)\,\lambda^{\frac{m-p}{p^2}}}{\left(\lambda + t^{\frac{p}{p-1}}\right)^{\frac{m}{p}-1}}.$$

*Moreover, choosing* $\beta(m,p) > 0$ *such that*

$$\int_{\mathbb{R}^m} \phi_\lambda^{p^*}(x)dx = 1 \tag{3.14}$$

*then*

$$K(m,p)^{-p} = \int_{\mathbb{R}^m} |\varphi_\lambda'(|x|)|^p \, dx$$

*and, by the standard calculus of variations, the extremal functions* $\phi_\lambda$ *obey the (nonlinear) Yamabe-type equation*

$${}^{\mathbb{R}^m}\Delta_p \phi_\lambda = -K(m,p)^{-p}\,\phi_\lambda^{p^*-1}. \tag{3.15}$$

**Lemma 3.11** (Comparisons for the Ricci tensor). *Let* $(M,g)$ *be a complete Riemannian manifold satisfying* $Ric_M \geq (m-1)c$ *for some* $c \in \mathbb{R}$*. Let* $r(x) = d(x,o)$ *for some origin* $o \in M$*. Then:*

*(i) Laplacian comparison.*

$$\Delta r \leq (m-1)\frac{\mathrm{sn}_c'(r(x))}{\mathrm{sn}_c(r(x))}$$

*pointwise on* $M \backslash \mathrm{cut}(o)$ *and in the distributional sense on all of* $M$*. Moreover, the equality holds on some ball* $B_{R_0}(o)$ *if and only if* $B_{R_0}(o)$ *is isometric to the corresponding ball* $\mathbb{B}_R$ *in the* $m$*-dimensional space form of constant curvature* $c$*.*

*(ii)* *Bishop-Gromov comparison*

$$R \longmapsto \frac{\operatorname{area}(\partial B_R(o))}{\operatorname{area}(\partial \mathbf{B}_R)} \text{ is decreasing}$$

*for every $R > 0$, where $\mathbf{B}_R$ is the ball in the spaceform $\mathbf{M}^m(c)$ of constant curvature $c$. In particular, $\forall R > 0$,*

$$(a)\ \operatorname{vol}(\partial B_R(o)) \le \operatorname{vol}(\partial \mathbf{B}_R), \qquad (b)\ \operatorname{vol}(B_R(o)) \le \operatorname{vol}(\mathbf{B}_R)$$

*and the equality holds for some $R_0 > 0$ if and only if $B_{R_0}(o)$ is isometric to $\mathbf{B}_{R_0}$.*

We are now in the position to give the

**Proof (of Theorem 3.9).** We transplant on $M$ the extremal function $\phi = \phi_1$ of $\mathbb{R}^m$ thus obtaining the radial $Lip_{loc}$ function $\Phi : M \to \mathbb{R}$ as

$$\Phi(x) := \varphi(r(x)).$$

Then $\nabla\Phi = \varphi'\nabla r$ and, recalling that $|\nabla r| = 1$, we can compute

$$\Delta_p \Phi = \left(|\varphi'|^{p-2}\varphi'\nabla r\right)' + |\varphi'|^{p-2}\varphi'\Delta r.$$

Since $\varphi$ is decreasing and, by the Laplacian comparison theorem,

$$\Delta r \le \frac{(m-1)}{r},$$

from the above equation we deduce

$$\Delta_p \Phi_\lambda \ge {}^{\mathbb{R}^m}\Delta_p \phi_\lambda.$$

Inserting in the Yamabe equation (3.15), we conclude that $\Phi$ satisfies the differential inequality

$$\Delta_p \Phi \ge -K(m,p)^{-p}\,\Phi^{p^*-1} \tag{3.16}$$

pointwise on $M\backslash \operatorname{cut}(o)$ and weakly on all of $M$.

We claim that

$$\begin{cases} \text{(i)} & \Phi \in L^{p^*}(M) \text{ with } \int_M \Phi^{p^*} \le 1, \\ \text{(ii)} & |\nabla\Phi| \in L^p(M), \end{cases} \tag{3.17}$$

and, therefore, that the vector field

$$X := \Phi\,|\nabla\Phi|^{p-2}\,\nabla\Phi$$

satisfies

$$\begin{array}{ll} \text{(i)} & \int_{B_{2R}\backslash B_R} |X| = o(R), \text{ as } R \to +\infty, \\ \text{(ii)} & (\operatorname{div} X)_- \in L^1(M). \end{array} \tag{3.18}$$

Suppose for the moment that we have already proved the claim and let us show how to complete the proof. According to (3.18) we can apply the Karp version of Stokes theorem and, using also (3.16), we obtain

$$0 = \int_M \operatorname{div} X \geq \int_M |\nabla \Phi|^p - K(m,p)^{-p} \int_M \Phi^{p^*}$$

i.e.

$$\int_M |\nabla \Phi|^p \leq K(m,p)^{-p} \int_M \Phi^{p^*}.$$

On the other hand, because of (3.17) we can use $\Phi$ as a test function in the Sobolev inequality and get

$$\int_M |\nabla \Phi|^p \geq K(m,p)^{-p} \left( \int_M \Phi^{p^*} \right)^{\frac{1}{p^*}}.$$

Combining these two inequalities, and using again (3.17) (i) and $p^* > 1$, we deduce

$$\int_M \Phi^{p^*} \leq \left( \int_M \Phi^{p^*} \right)^{\frac{1}{p^*}} \leq \int_M \Phi^{p^*},$$

i.e.

$$\int_M \Phi^{p^*} = 1 = \int_{\mathbb{R}^m} \phi^{p^*}.$$

Whence, using the co-area formula to compute the two integrals we obtain

$$\int_0^{+\infty} \varphi(R)^{p^*} \{\operatorname{area}(\partial B_R) - \operatorname{area}(\partial \mathbf{B}_R)\}\, dR = 0.$$

Since, by volume comparison, $\operatorname{area}(\partial B_R) \leq \operatorname{area}(\partial \mathbf{B}_R)$ we must conclude

$$\operatorname{area}(\partial B_R) = \operatorname{area}(\partial \mathbf{B}_R),\ \forall R > 0,$$

and the rigidity part of the Theorem follows from the equality case in the Bishop-Gromov comparison.

It remains to show the validity of the claimed properties (3.17) and (3.18). By the co-area formula and the area comparison, we have

$$\begin{aligned}
\int_M \Phi^{p^*} &= \int_0^{+\infty} \int_{\partial B_R} \varphi^{p^*}(R)\, dR \\
&= \int_0^{+\infty} \varphi^{p^*}(R) \operatorname{area}(\partial B_R)\, dR \\
&\leq \int_0^{+\infty} \varphi^{p^*}(R) \operatorname{area}(\partial \mathbf{B}_R)\, dR \\
&= \int_{\mathbb{R}^m} \phi^{p^*} \\
&= 1,
\end{aligned}$$

proving (3.17) (i). The proof of (3.17) (ii) is completely similar. Now consider the vector field $X = \Phi |\nabla\Phi|^{p-2} \nabla\Phi$. Using Hölder inequality and (3.17) (ii), we obtain

$$\int_{B_{2R}\backslash B_R} |X| = \int_{B_{2R}\backslash B_R} \Phi |\nabla\Phi|^{p-1} \leq \left(\int_{B_{2R}\backslash B_R} \Phi^p\right)^{\frac{1}{p}} \left(\int_{B_{2R}\backslash B_R} |\nabla\Phi|^p\right)^{\frac{p-1}{p}}$$
$$\leq C\left(\int_{B_{2R}\backslash B_R} \Phi^p\right)^{\frac{1}{p}}. \tag{3.19}$$

On the other hand, applying Hölder inequality with exponents $m/(m-p)$ and $p/(m-p)$, we have

$$\left(\int_{B_{2R}\backslash B_R} \Phi^p\right)^{\frac{1}{p}} \leq \left(\int_{B_{2R}\backslash B_R} \Phi^{p^*}\right)^{\frac{1}{p^*}} \mathrm{vol}\,(B_{2R}\backslash B_R)^{\frac{1}{m}},$$

where, by (3.17) (i),

$$\left(\int_{B_{2R}\backslash B_R} \Phi^{p^*}\right)^{\frac{1}{p^*}} = o(1) \tag{3.20}$$

and, by Bishop-Gromov comparison,

$$\mathrm{vol}\,(B_{2R}\backslash B_R)^{\frac{1}{m}} = O(R) \tag{3.21}$$

as $R \to +\infty$. Inserting (3.20) and (3.21) into (3.19) proves the validity of (3.18) (i).

Finally

$$\begin{aligned}\mathrm{div}\, X &= |\nabla\Phi|^p + \Phi\Delta_p\Phi \\ &\geq -K(m,p)^{-p}\,\Phi^{p^*} \in L^1(M)\end{aligned}$$

and this proves (3.18) (ii). ■

# Appendix A

# On the optimality of cut-off functions

Let $(M, g)$ be a complete Riemannian manifold. Then, to an exhaustion of $M$ by relatively compact geodesic balls $B_R(o)$ there corresponds a family of radial cut-off functions $\varphi_R \in C_c^\infty(M)$ satisfying the following requirements

| | |
|---|---|
| (a) $0 \le \varphi_R \le 1$, | (b) $\varphi_R = 1$, on $B_R(o)$, |
| (c) $\varphi_R = 0$ on $M \backslash B_{2R}(o)$, | (d) $\|\nabla \varphi_R\|_\infty \le \frac{2}{R}$ on $M$. |

This family was constructed in Theorem 2.28 using the piecewise linear functions (2.10) and can be smoothen-out. Since the improvement of many analytic and geometric results involving $L^p$ conditions relies on the decay condition (d) it is natural to enquire whether a better choice is possible. The answer is no, as we shall see momentarily

More precisely, suppose you are given an exhaustion of $M$ by geodesic balls $B_R(o)$ and a corresponding family $\varphi_R \in C_c^\infty(M)$ satisfying conditions (a)–(c). We ask if $\varphi_R$ can be chosen so to satisfy

$$\|\nabla \varphi_R\|_\infty \le \frac{\varepsilon(R)}{R},$$

where $\varepsilon(R) \searrow 0$, as $R \to +\infty$.

Let us mention that, curiously enough, the answer on the Euclidean space $\mathbb{R}^m$ is related to capacitary considerations. Indeed, let $F(r) \ge 0$ be any non-increasing function satisfying

$$\sup_{\partial \mathbf{B}_R} |\nabla \varphi_R| \le F(R).$$

Consider the condenser $\mathbf{E}_R = \left(\overline{\mathbf{B}}_R, \mathbf{B}_{2R}\right)$ and note that its equilibrium potential is given by

$$\mathbf{u}(r) = \frac{\int_r^{2R} \frac{dt}{t^{m-1}}}{\int_R^{2R} \frac{dt}{t^{m-1}}}.$$

Thus, using also the divergence theorem,

$$\operatorname{cap}(\mathbf{E}_R) = -\int_{\partial \mathbf{B}_R} \langle \nabla \mathbf{u}, \nabla r\rangle = -\mathbf{u}'(R) \operatorname{area}(\partial \mathbf{B}_R) \geq C R^{m-2},$$

for some universal constant $C = C(m) > 0$. On the other hand, by definition of capacity,

$$\operatorname{cap}(\mathbf{E}_R) \leq \int_{\mathbf{B}_{2R} \setminus \mathbf{B}_R} |\nabla \varphi_R|^2 \leq F(R)^2 \operatorname{area}(\mathbf{B}_{2R}) \leq C F(R)^2 R^m,$$

for some other absolute constant $C = C(m) > 0$. It follows that

$$F(R) \geq \frac{C}{R}$$

and, in particular,

$$\varepsilon(R) \geq C = C(m) > 0.$$

The same conclusion actually holds on any complete manifold $(M, g)$. Indeed, take a unit-speed minimizing geodesic $\gamma : [0, 2R] \to M$ issuing from $\gamma(0) = o$. Then $\gamma(r) \in \partial B_r(o)$ and we have

$$F(R) \geq \sup_{B_{2R} \setminus B_R} |\nabla \varphi| \geq \sup_{[R,2R]} |\nabla \varphi| \circ \gamma \geq \sup_{[R,2R]} |\langle \nabla \varphi, \dot{\gamma}\rangle| = \sup_{[R,2R]} \left| \frac{d}{dt} \varphi \circ \gamma \right|.$$

Since a *Lip* function $f : [R, 2R] \to [0, 1]$ such that $f(R) = 1$ and $f(2R) = 0$ must satisfy

$$\sup_{[R,2R]} \left| \frac{d}{dt} f \right| \geq \frac{C}{R},$$

the asserted property follows.

We conclude by remarking that there are situations where the standard choice of radial cut-off functions does not give the best possible result. An example can be found in [55], see Remark 9 there.

# Appendix B

# The strong comparison principle for graphs

This section aims to prove the following comparison result for second order, elliptic operators in divergence form. The proof is adapted from a classical paper by R. Schoen, [42].

**Theorem B.1.** *Suppose you are given the non-linear operator*

$$L(f) = \operatorname{div} \Phi(\nabla f),$$

*where $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth vector field satisfying the following conditions:*

*(a) $\Phi$ is symmetric, i.e.,*

$$\frac{\partial \Phi_j}{\partial u_i} = \frac{\partial \Phi_i}{\partial u_j}. \tag{B.1}$$

*(b) The matrix $[\partial \Phi_i / \partial u_j]$ is positive definite, i.e.,*

$$\sum_{i,j} \frac{\partial \Phi_j}{\partial u_i} \xi_i \xi_j \geq 0, \text{ with } = 0 \Leftrightarrow \xi = 0. \tag{B.2}$$

*Then, the strong comparison principle holds for $L$. Namely let $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be $C^2$ functions on a domain $\Omega$ satisfying*

$$L(f) - L(g) \geq 0, \text{ on } \Omega.$$

*Then, $f - g$ cannot attain an interior maximum in $\Omega$ unless $f - g$ is constant.*

**Example B.2.** *The mean curvature operator corresponds to the choice*

$$\Phi(u) = \frac{u}{\sqrt{1+|u|^2}}.$$

*Since*

$$\frac{\partial \Phi_i}{\partial u_j} = \left(\delta_{ij} - \frac{u_i u_j}{1+|u|^2}\right)\frac{1}{\sqrt{1+|u|^2}},$$

*conditions (B.1) and (B.2) are satisfied. Therefore, according to Theorem B.1, the strong comparison principle holds for the mean curvature operator.*

**Proof.** By the fundamental theorem of calculus,

$$\begin{aligned}\Phi(u) - \Phi(v) &= \int_0^1 \frac{d}{dt}\Phi(tu + (1-t)v)\,dt \qquad \text{(B.3)}\\ &= \sum_k (u_k - v_k)\int_0^1 \frac{\partial \Phi}{\partial u_k}(tu + (1-t)v)\,dt.\end{aligned}$$

Using (B.3) and after some direct computations, we obtain

$$\begin{aligned}L(f) - L(g) = &\sum_k \left\{\sum_{i,j}\frac{\partial^2 g}{\partial x_i \partial x_j}\int_0^1 \frac{\partial^2 \Phi_j}{\partial u_i \partial u_k}(t\nabla f + (1-t)\nabla g)\,dt\right\}\frac{\partial(f-g)}{\partial x_k}\\ &+ \sum_{i,j}\left\{\int_0^1 \sum_k \frac{\partial(f-g)}{\partial x_k}\frac{\partial^2 \Phi_j}{\partial u_i \partial u_k}(t\nabla f + (1-t)\nabla g)\,tdt\right.\\ &\left.+ \int_0^1 \frac{\partial \Phi_j}{\partial u_i}(t\nabla f + (1-t)\nabla g)\right\}\frac{\partial^2(f-g)}{\partial x_i \partial x_j}. \qquad \text{(B.4)}\end{aligned}$$

Now, observe that

$$\sum_k \frac{\partial(f-g)}{\partial x_k}\frac{\partial^2 \Phi_j}{\partial u_i \partial u_k}(t\nabla f + (1-t)\nabla g) = \frac{d}{dt}\frac{\partial \Phi_j}{\partial u_i}(t\nabla f + (1-t)\nabla g). \qquad \text{(B.5)}$$

Therefore, integrating by parts,

$$\begin{aligned}&\int_0^1 \sum_k \frac{\partial(f-g)}{\partial x_k}\frac{\partial^2 \Phi_j}{\partial u_i \partial u_k}(t\nabla f + (1-t)\nabla g)\,tdt \qquad \text{(B.6)}\\ &= \frac{\partial \Phi_j}{\partial u_i}(\nabla f) - \int_0^1 \frac{\partial \Phi_j}{\partial u_i}(t\nabla f + (1-t)\nabla g)\,dt.\end{aligned}$$

Inserting (B.6) into (B.4) gives

$$L(f) - L(g) = \mathcal{L}(f-g), \qquad \text{(B.7)}$$

where

$$\mathcal{L} = \sum_{i,j} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_k b_k \frac{\partial}{\partial x_k},$$

and

$$a_{ij} = \frac{\partial \Phi_j}{\partial u_i} (\nabla f) \tag{B.8}$$

$$b_k = \sum_{i,j} \frac{\partial^2 g}{\partial x_i \partial x_j} \int_0^1 \frac{\partial^2 \Phi_j}{\partial u_i \partial u_k} (t \nabla f + (1-t) \nabla g) \, dt.$$

Since $\mathcal{L}$ is linear, the conclusion follows immediately from the usual strong maximum principle for second order, linear elliptic operators; see e.g. [10].
■

# Appendix C

# The maximum of subsolutions is a subsolution

Very often it is useful to produce a subsolution (respectively, a supersolution) of a given PDE involving an elliptic operator in divergence form, by taking the maximum (respectively the minimum) of subsolutions (respectively of supersolutions). Think, for instance, of the proof of the Liouville result by Yau or the proof of the Ahlfors characterization of parabolicity. To the best of our knowledge, the most general result in this direction appears in a paper by V.K. Le, [25]. We shall provide a proof of this result in the simplified form which is needed in our applications. The arguments are adapted from Le's paper.

For the sake of completeness, recall that given $u \in W^{1,p}_{loc}(M) \cap C^0(M)$ its positive part $u_+ = \max(u, 0)$ has the same regularity and

$$\nabla u_+ = \begin{cases} \nabla u, & \text{if } u > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Recall also that, for any $\alpha \in Lip_{loc}(\mathbb{R})$ we have $\alpha \circ u \in W^{1,p}_{loc}(M) \cap C^0(M)$ and the usual chain rule holds, namely,

$$\nabla(\alpha \circ u) = \alpha'(u)\nabla u, \text{ a.e.}$$

**Theorem C.1.** *Let $(M, g)$ be a Riemannian manifold and $\Phi : TM \to TM$ an endomorphism satisfying*

$$|\Phi(\xi)| \leq A|\xi|^{p-1} \tag{C.1}$$

*and*

$$\langle \Phi(\xi), \xi \rangle \geq 0, \tag{C.2}$$

*for some constants* $A > 0$, $p > 1$ *and for every vector field* $\xi$. *Consider the second order, elliptic operator*

$$L(u) = \operatorname{div} \Phi(\nabla u) + q(x) u, \tag{C.3}$$

*where* $q \in L^{\infty}_{loc}(M)$. *If* $u \in W^{1,p}_{loc}(M) \cap C^0(M)$ *satisfy* $L(u) \geq 0$ *in the sense of distributions, then* $u_+ = \max(u, 0) \in W^{1,p}_{loc}(M) \cap C^0(M)$ *solves the same inequality.*

**Remark C.2.** *The* $\varphi$*-Laplacian corresponds to the choice* $\Phi(\xi) = |\xi|^{-1} \varphi(|\xi|) \xi$.

**Proof.** Take any function $\lambda \in C^{\infty}(\mathbb{R})$ satisfying

$$\begin{cases} 0 \leq \lambda(t) \leq 1, \\ \lambda'(t) \geq 0, \\ \lambda(t) = 0, \ \forall t \leq 0, \\ \lambda(t) = 1, \ \forall t \geq 1, \end{cases}$$

and define

$$\lambda_n(t) = \lambda(nt),$$

so that $\lambda_n(t) = 0$ for $t \leq 0$, $\lambda_n(t) = 1$ if $t \geq 1/n$ and $\lambda_n'(t) \leq Cn$ for every $t \in \mathbb{R}$ and for some universal constant $C > 0$. In particular, $\lambda_n(t) \to \chi_{(0,+\infty)}$ pointwise on $\mathbb{R}$. Having fixed $0 \leq \rho \in C^{\infty}_c(M)$ let $\Omega \subset\subset M$ be a smooth, relatively compact domain with $\operatorname{supp}(\rho) \subset\subset \Omega$. Since $u_+ \in W^{1,p}(\Omega)$, by standard density results, there exists a sequence $\{u_n\} \in C^{\infty}_c(M)$ such that

$$u_n^+ \to u_+ \text{ in } W^{1,p}(\Omega).$$

In particular, up to passing to a subsequence (still denoted by $u_n$) we can assume that

$$u_n^+ \to u_+ \text{ a.e. in } \Omega,$$

and

$$\|u_n^+ - u_+\|_{W^{1,p}(\Omega)} \leq \frac{1}{n^2}. \tag{C.4}$$

Using the $W^{1,p}_0(M)$ test function

$$0 \leq \phi = \lambda_n(u_n^+) \rho$$

in the distributional definition of $L(u) \geq 0$ we get

$$\begin{aligned} &- \int_M \langle \Phi(\nabla u), \nabla \rho \rangle \lambda_n(u_n^+) - \int_M \langle \Phi(\nabla u), \nabla u_n^+ \rangle \lambda_n'(u_n^+) \rho \\ &+ \int_M q(x) u \lambda_n(u_n^+) \rho =: a_n + b_n + c_n \geq 0. \end{aligned}$$

We shall prove that

$$\lim_{n\to+\infty} c_n = \int_M q(x)\, u_+ \rho, \tag{C.5}$$

next

$$\lim_{n\to+\infty} a_n = -\int_M \langle \Phi(\nabla u_+), \nabla\rho\rangle \tag{C.6}$$

and, finally,

$$\limsup_{n\to+\infty} b_n \le 0. \tag{C.7}$$

Obviously, this will complete the proof.

Equation (C.5) follows immediately from dominated convergence because

$$u_n^+ \to u_+ \text{ a.e. in } \Omega$$

and, hence,

$$\lambda_n(u_n^+) \to \chi_{\{u>0\}} \text{ a.e. in } \Omega.$$

Similarly, in order to obtain (C.6) we note that

$$|\lambda_n(u_n^+)\langle \Phi(\nabla u), \nabla\rho\rangle| \le A|\nabla u|^{p-1}|\nabla\rho| \in L^1(\Omega).$$

Therefore, we can apply the dominated convergence theorem and deduce

$$a_n \to -\int_{\{u>0\}\cap\Omega} \langle \Phi(\nabla u), \nabla\rho\rangle = -\int_{\{u>0\}} \langle \Phi(\nabla u), \nabla\rho\rangle = \int_M \langle \Phi(\nabla u_+), \nabla\rho\rangle.$$

It remains to prove (C.7). To this end, we first observe the estimate

$$\begin{aligned} b_n &= -\int_M \langle \Phi(\nabla u), \nabla u_n^+\rangle \lambda_n'(u_n)\rho \\ &= -\int_M \langle \Phi(\nabla u), \nabla u_n^+ - \nabla u_+\rangle \lambda_n'(u_n^+)\rho - \int_M \langle \Phi(\nabla u), \nabla u_+\rangle \lambda_n'(u_n^+)\rho \\ &\le \int_M |\Phi(\nabla u)||\nabla u_n^+ - \nabla u_+|\lambda_n'(u_n^+)\rho - \int_{\{u>0\}} \langle \Phi(\nabla u), \nabla u_+\rangle \lambda_n'(u_n^+)\rho \\ &=: b_n^{(1)} + b_n^{(2)}. \end{aligned}$$

Now, by the ellipticity condition (C.2) and the fact that $\rho, \lambda_n' \ge 0$ we have

$$b_n^{(2)} = -\int_{\{u>0\}} \langle \Phi(\nabla u_+), \nabla u_+\rangle \lambda_n'(u_n^+)\rho \le 0.$$

On the other hand, using Hölder inequality and (C.4),

$$\begin{aligned} b_n^{(1)} &\le \sup_M \lambda_n'(u_n^+) \int_M |\Phi(\nabla u)||\nabla u_n^+ - \nabla u_+|\rho \\ &\le ACn \int_M |\nabla u|^{p-1}|\nabla u_n^+ - \nabla u^+|\rho \\ &\le ACn \|\nabla u\|_{L^p(\Omega)}^{p-1} \|\nabla u_n^+ - \nabla u_+\|_{L^p(\Omega)} \\ &\le \frac{AC}{n} \|\nabla u\|_{L^p(\Omega)}^{p-1} \to 0. \end{aligned}$$

It follows that

$$\limsup_{n\to+\infty} b_n \leq \limsup_{n\to+\infty} b_n^{(1)} = 0,$$

as claimed. ■

# Bibliography


[1] K. Akutagawa, *Yamabe metrics of positive scalar curvature and conformally flat manifolds.* Differential Geom. Appl. 4 (1994), no. 3, 239–258.

[2] G. Carron, *Inégalités isopérimétriques et inégalités de Faber-Krahn.* Séminaire de Théorie Spectrale et Géométrie, No. 13, Année 1994–1995, 63–66.

[3] G. Carron, *Inégalité de Sobolev et volume asymptotique.* Ann. Fac. Sci. Toulouse Math. **21** (2012), 151–172.

[4] Q. Chen, *Liouville theorem for harmonic maps with potential.* Manuscripta Math. **95** (1998), 507–517.

[5] S.Y. Cheng, S.T. Yau, *Differential equations on Riemannian manifolds and their geometric applications.* Comm. Pure Appl. Math. **28** (1975), 333–354.

[6] J. Eells, James, J.H. Sampson, *Harmonic mappings of Riemannian manifolds.* Amer. J. Math. **86** (1964), 109–160.

[7] P. Gaffney, *A special Stokes's theorem for complete Riemannian manifolds.* Ann. of Math. (2) **60**, (1954), 140–145.

[8] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds.* Bull. Amer. Math. Soc. (N.S.) **36** (1999), 135–249.

[9] V. Gol'dshtein, M. Troyanov, *The Kelvin-Nevanlinna-Royden criterion for $p$-parabolicity.* Math. Z. **232** (1999), 607–619.

[10] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[11] V. Gol'dshtein, M. Troyanov, *Axiomatic theory of Sobolev spaces.* Expo. Math. **19** (2001) 289–336.

[12] V. Gol'dshtein, M. Troyanov, *Capacities in metric spaces.* Integral Equations Operator Theory **44** (2002), 212–242.

[13] V. Gol'dshtein, M. Troyanov, *Sobolev inequalities for differential forms and $L^{q,p}$ cohomology.* Jour. Geom. Anal. **16** (2006), 597–631.

[14] R. Greene, H.H. Wu, *Function theory on manifolds which possess a pole.* Lecture Notes in Mathematics, **699**. Springer, Berlin, 1979.

[15] R. Greene, H.H. Wu, *$C^\infty$ approximations of convex, subharmonic and plurisubharmonic functions.* Annales scientifiques de l'E.N.S. **12** (1979), 47–84.

[16] P. Hartman, *On homotopic harmonic maps.* Canad. J. Math. **19** (1967), 673–687.

[17] E. Hebey, *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities.* CIMS Lecture Notes (1999). Courant Institute of Mathematical Sciences.

[18] I. Holopainen, *Nonlinear potential theory and quasiregular mappings on Riemannian manifolds.* Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes, **74** (1990), 45 pp.

[19] I. Holopainen, Rough isometries and $p$-harmonic functions with finite Dirichlet integral. Rev. Mat. Iberoamericana **10** (1994), 143–176.

[20] I. Holopainen, *Quasiregular mappings and the $p$-Laplace operator.* Contemp. Math., **338** Amer. Math. Soc., Providence, RI, 2003.

[21] I. Holopainen, S. Pigola, G. Veronelli, *Global comparison principles for the $p$-Laplace operator on Riemannian manifolds.* Potential Anal. **34** (2011), 371–384.

[22] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear potential theory of degenerate elliptic equations.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[23] L. Karp, *On Stokes' theorem for noncompact manifolds.* Proc. Amer. Math. Soc. **82** (1981), 487–490.

[24] S. Kawai, *$p$-harmonic maps and convex functions.* Geom. Dedicata **74** (1999), 261–265.

[25] V.K. Le, *On some equivalent properties of sub- and supersolutions in second order quasilinear elliptic equations.* Hiroshima Math. J. **28** (1998), 373–380.

[26] M. Ledoux, *On manifolds with non-negative Ricci curvature and Sobolev inequalities.* Comm. Anal. Geom. **7** (1999), 347–353.

[27] M. Ledoux, *The geometry of Markov diffusion generators.* Ann. Fac. Sci. Toulouse Math. (6) **9** (2000), 305–366.

[28] P. Li, R. Schoen, *$L^p$ and mean value properties of subharmonic functions on Riemannian manifolds.* Acta Math. **153** (1984), 279–301.

[29] P. Li, L.F. Tam, *Harmonic functions and the structure of complete manifolds.* J. Diff. Geom. **35** (1992), 359–383.

[30] G. M. Lieberman, *Boundary regularity for degenerate elliptic equations.* Non. Lin. Anal. **12** (1988),1203–1219.

[31] T. Lyons, D. Sullivan, *Function theory, random paths and covering spaces.* J. Differential Geom. **19** (1984), 299–323.

[32] L. Mari, D. Valtorta, *On the equivalence of stochastic completeness, Liouville and Khas'minskii condition in linear and nonlinear setting.* Trans. Amer. Math. Soc. **365** (2013), 4699–4727.

[33] N. Nakauchi, *A Liouville type theorem for p-harmonic maps.* Osaka J. Math. **35** (1998), 303–312.

[34] S. Pigola, A.G. Setti, M. Troyanov, *The connectivity at infinity of a manifold and $L^{q,p}$-Sobolev inequalities.* Exposition. Math., in press.

[35] S. Pigola, M. Rigoli, A.G. Setti, *Vanishing and finiteness results in geometric analysis. A generalization of the Bochner technique.* Progress in Mathematics, 266. Birkhäuser Verlag, Basel, 2008.

[36] S. Pigola, M. Rigoli, A.G. Setti, *Constancy of p-harmonic maps of finite q-energy into non-positively curved manifolds.* Math. Z. **258** (2008), 347–362.

[37] S. Pigola, M. Rigoli, A.G. Setti, *Some non-linear function theoretic properties of Riemannian manifolds.* Rev. Mat. Iberoam. **22** (2006), 801–831.

[38] S. Pigola, G. Veronelli, *Remarks on $L^p$-vanishing results in geometric analysis.* International. Jour. Math. **23** (2012), no. 1, 1250008, 18 pp.

[39] S. Pigola, G. Veronelli, *Lower volume estimates and Sobolev inequalities.* Proc. Amer. Math. Soc. **138** (2010), 4479–4486.

[40] S. Pigola, G. Veronelli, *On the homotopy class of maps with finite p-energy into non-positively curved manifolds.* Geom. Dedicata **143** (2009), 109–116.

[41] M. Rigoli, A.G. Setti, *Liouville type theorems for $\varphi$-subharmonic functions.* Rev. Mat. Iberoamericana **17** (2001), no. 3, 471–520.

[42] R. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces.* J. Differential Geom. **18** (1983), 791–809.

[43] R. Strichartz, *Analysis of the Laplacian on the complete Riemannian manifold.* J. Funct. Anal. **52** (1983), 48–79.

[44] R. Schoen, S.-T. Yau, *Harmonic maps and the topology of stable hypersurfaces and manifolds with non-negative ricci curvature.* Comm. Math. Helv. **51** (1976), 333–341.

[45] R. Schoen, S.-T. Yau, *Compact group actions and the topology of manifolds with nonpositive curvature.* Topology **18** (1979), 361–380.

[46] R. Schoen, S.-T. Yau, *Lectures on harmonic maps.* Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997.

[47] K. Takegoshi, *A maximum principle for p-harmonic maps with $L^q$ finite energy.* Proc. Am. Math. Soc. **126** (1998), 3749–3753.

[48] P. Tolksdorf, *Regularity of a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126–150.

[49] M. Troyanov, *Parabolicity of manifolds.* Siberian Adv. Math. **9** (1999), no. 4, 125–150.

[50] M. Troyanov, *Solving the p-Laplacian on manifolds.* Proc. Amer. Math. Soc. **128** (2000), 541–545.

[51] D. Valtorta, *Potenziali di Evans su varietà paraboliche.* Master Thesis. http://arxiv.org/abs/1101.2618.

[52] D. Valtorta, *Reverse Khas'minskii condition.* Math. Z. **270** (2012), 165–177.

[53] G. Veronelli, *On p-harmonic maps and convex functions.* Manuscripta Math. **131** (2010), no. 3-4, 537–546.

[54] G. Veronelli, *A general comparison theorem for p-harmonic maps in homotopy class.* Jour. Math. Anal. Appl. **391** (2012), 335–349.

[55] D. Valtorta, G. Veronelli, *Stokes' theorem, volume growth and parabolicity.* Tohoku Math. J. **63** (2011), 397–412.

[56] W.S. Wei, *Representing homotopy groups and spaces of maps by p-harmonic maps.* Indiana Univ. Math. J. **47** (1998), 625–670.

[57] C. Xia, *Complete manifolds with nonnegative Ricci curvature and almost best Sobolev constant.* Illinois J. Math. **45** (2001), 1253–1259.

[58] S.T. Yau, *Harmonic functions on complete Riemannian manifolds.* Comm. Pure Appl. Math. **28** (1975), 201–228.

Stefano Pigola
Dipartimento di Scienza e Alta Tecnologia, Università dell'Insubria
Via Valleggio 11, 22100 Como, Italy
stefano.pigola@uninsubria.it
`http://www.dfm.uninsubria.it/pigola`

Alberto G. Setti
Dipartimento di Scienza e Alta Tecnologia, Università dell'Insubria
Via Valleggio 11, 22100 Como, Italy
alberto.setti@uninsubria.it
`http://www.dfm.uninsubria.it/setti`